ARML Competition 2009

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1 Individual Problems

Problem 1. Let p be a prime number. If p years ago, the ages of three children formed a geometric sequence with a sum of p and a common ratio of 2, compute the sum of the children's current ages.

Problem 2. Define a *reverse prime* to be a positive integer N such that when the digits of N are read in reverse order, the resulting number is a prime. For example, the numbers 5, 16, and 110 are all reverse primes. Compute the largest two-digit integer N such that the numbers $N, 4 \cdot N$, and $5 \cdot N$ are all reverse primes.

Problem 3. Some students in a gym class are wearing blue jerseys, and the rest are wearing red jerseys. There are exactly 25 ways to pick a team of three players that includes at least one player wearing each color. Compute the number of students in the class.

Problem 4. Point P is on the hypotenuse $\overline{EN}$ of right triangle BEN such that $\overline{BP}$ bisects $\angle EBN$. Perpendiculars $\overline{PR}$ and $\overline{PS}$ are drawn to sides $\overline{BE}$ and $\overline{BN}$, respectively. If $EN = 221$ and $PR = 60$, compute $\frac{1}{BE} + \frac{1}{BN}$.

Problem 5. Compute all real values of x such that $\log_2(\log_2 x) = \log_4(\log_4 x)$.

Problem 6. Let k be the least common multiple of the numbers in the set $\mathcal{S} = \{1, 2, \ldots, 30\}$. Determine the number of positive integer divisors of k that are divisible by exactly 28 of the numbers in the set $\mathcal{S}$.

Problem 7. Let A and B be digits from the set $\{0, 1, 2, \ldots, 9\}$. Let r be the two-digit integer $\underline{A}\ \underline{B}$ and let s be the two-digit integer $\underline{B}\ \underline{A}$, so that r and s are members of the set $\{00, 01, \ldots, 99\}$. Compute the number of ordered pairs (A, B) such that $|r - s| = k^2$ for some integer k.

Problem 8. For $k \geq 3$, we define an ordered k-tuple of real numbers $(x_1, x_2, \ldots, x_k)$ to be *special* if, for every i such that $1 \leq i \leq k$, the product $x_1 \cdot x_2 \cdot \ldots \cdot x_k = x_i^2$. Compute the smallest value of k such that there are at least 2009 distinct special k-tuples.

Problem 9. A cylinder with radius r and height h has volume 1 and total surface area 12. Compute $\frac{1}{r} + \frac{1}{h}$.

Problem 10. If $6 \tan^{-1} x + 4 \tan^{-1}(3x) = \pi$, compute x^2.

2 Individual Answers

Answer 1. 28

Answer 2. 79

Answer 3. 7

Answer 4. $\frac{1}{60}$

Answer 5. $\sqrt{2}$

Answer 6. 23

Answer 7. 42

Answer 8. 12

Answer 9. 6

Answer 10. $\frac{15-8\sqrt{3}}{33}$

3 Individual Solutions

Problem 1. Let p be a prime number. If p years ago, the ages of three children formed a geometric sequence with a sum of p and a common ratio of 2, compute the sum of the children's current ages.

Solution 1. Let x, $2x$, and $4x$ be the ages of the children p years ago. Then $x + 2x + 4x = p$, so $7x = p$. Since p is prime, $x = 1$. The sum of the children's current ages is therefore $(1 + 7) + (2 + 7) + (4 + 7) = 28$.

Problem 2. Define a *reverse prime* to be a positive integer N such that when the digits of N are read in reverse order, the resulting number is a prime. For example, the numbers 5, 16, and 110 are all reverse primes. Compute the largest two-digit integer N such that the numbers $N, 4 \cdot N$, and $5 \cdot N$ are all reverse primes.

Solution 2. Because $N < 100$, $5 \cdot N < 500$. Since no primes end in 4, it follows that $5 \cdot N < 400$, hence $N \leq 79$. The reverses of $5 \cdot 79 = 395$, $4 \cdot 79 = 316$, and 79 are 593, 613, and 97, respectively. All three of these numbers are prime, thus 79 is the largest two-digit integer N for which N, $4 \cdot N$, and $5 \cdot N$ are all reverse primes.

Problem 3. Some students in a gym class are wearing blue jerseys, and the rest are wearing red jerseys. There are exactly 25 ways to pick a team of three players that includes at least one player wearing each color. Compute the number of students in the class.

Solution 3. Let r and b be the number of students wearing red and blue jerseys, respectively. Then either we choose two blues and one red or one blue and two reds. Thus

$$\begin{aligned} & \binom{b}{2}\binom{r}{1} + \binom{b}{1}\binom{r}{2} = 25 \\ \Rightarrow \quad & \frac{rb(b-1)}{2} + \frac{br(r-1)}{2} = 25 \\ \Rightarrow \quad & rb((r-1)+(b-1)) = 50 \\ \Rightarrow \quad & rb(r+b-2) = 50. \end{aligned}$$

Now because r, b, and $r + b - 2$ are positive integer divisors of 50, and $r, b \geq 2$, we have only a few possibilities to check. If $r = 2$, then $b^2 = 25$, so $b = 5$; the case $r = 5$ is symmetric. If $r = 10$, then $b(b + 8) = 5$, which is impossible. If $r = 25$, then $b(b + 23) = 2$, which is also impossible. So $\{r, b\} = \{2, 5\}$, and $r + b = 7$.

Problem 4. Point P is on the hypotenuse $\overline{EN}$ of right triangle BEN such that $\overline{BP}$ bisects $\angle EBN$. Perpendiculars $\overline{PR}$ and $\overline{PS}$ are drawn to sides $\overline{BE}$ and $\overline{BN}$, respectively. If $EN = 221$ and $PR = 60$, compute $\frac{1}{BE} + \frac{1}{BN}$.

Solution 4. We observe that $\frac{1}{BE} + \frac{1}{BN} = \frac{BE+BN}{BE \cdot BN}$. The product in the denominator suggests that we compare areas. Let $[BEN]$ denote the area of $\triangle BEN$. Then $[BEN] = \frac{1}{2}BE \cdot BN$, but because $PR = PS = 60$, we can also write $[BEN] = [BEP] + [BNP] = \frac{1}{2} \cdot 60 \cdot BE + \frac{1}{2} \cdot 60 \cdot BN$. Therefore $BE \cdot BN = 60(BE + BN)$, so $\frac{1}{BE} + \frac{1}{BN} = \frac{BE+BN}{BE \cdot BN} = \frac{1}{60}$. Note that this value does not depend on the length of the hypotenuse $\overline{EN}$; for a given location of point P, $\frac{1}{BE} + \frac{1}{BN}$ is invariant.

Alternate Solution: Using similar triangles, we have $\frac{ER}{PR} = \frac{PS}{SN} = \frac{BE}{BN}$, so $\frac{BE-60}{60} = \frac{60}{BN-60} = \frac{BE}{BN}$ and $BE^2 + BN^2 = 221^2$. Using algebra, we find that $BE = 204$, $BN = 85$, and $\frac{1}{204} + \frac{1}{85} = \frac{1}{60}$.

Problem 5. Compute all real values of x such that $\log_2(\log_2 x) = \log_4(\log_4 x)$.

Solution 5. If $y = \log_a(\log_a x)$, then $a^{a^y} = x$. Let $y = \log_2(\log_2 x) = \log_4(\log_4 x)$. Then $2^{2^y} = 4^{4^y} = (2^2)^{(2^2)^y} = 2^{2^{2y+1}}$, so $2y+1 = y$, $y = -1$, and $x = \sqrt{2}$.
(This problem is based on one submitted by ARML alum James Albrecht, 1986-2007.)

Alternate Solution: Raise 4 (or 2^2) to the power of both sides to get $(\log_2 x)^2 = \log_4 x$. By the change of base formula, $\frac{(\log x)^2}{(\log 2)^2} = \frac{\log x}{2 \log 2}$, so $\log x = \frac{\log 2}{2}$, thus $x = 2^{1/2} = \sqrt{2}$.

Alternate Solution: Let $x = 4^a$. The equation then becomes $\log_2(2a) = \log_4 a$. Raising 4 to the power of each side, we get $4a^2 = a$. Since $a \neq 0$, we get $4a = 1$, thus $a = \frac{1}{4}$ and $x = \sqrt{2}$.

Problem 6. Let k be the least common multiple of the numbers in the set $\mathcal{S} = \{1, 2, \ldots, 30\}$. Determine the number of positive integer divisors of k that are divisible by exactly 28 of the numbers in the set $\mathcal{S}$.

Solution 6. We know that $k = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$. It is not difficult to see that the set $\mathcal{T}_1 = \left\{\frac{k}{2}, \frac{k}{3}, \frac{k}{5}, \frac{k}{17}, \frac{k}{19}, \frac{k}{23}, \frac{k}{29}\right\}$ comprises all divisors of k that are divisible by exactly 29 of the numbers in the set $\mathcal{S}$. Let $\mathcal{P} = \{2, 3, 5, 17, 19, 23, 29\}$. Then $\mathcal{T}_2 = \left\{\frac{k}{p_1 p_2}, \text{ where } p_1 \text{ and } p_2 \text{ are distinct elements of } \mathcal{P}\right\}$ consists of divisors of k that are divisible by exactly 28 of the numbers in the set $\mathcal{S}$. There are $\binom{7}{2} = 21$ elements in $\mathcal{T}_2$.

Furthermore, note that $\frac{k}{7}$ is only divisible by 26 of the numbers in $\mathcal{S}$ (since it is not divisible by 7, 14, 21, or 28) while $\frac{k}{11}$ and $\frac{k}{13}$ are each divisible by 28 of the numbers in $\mathcal{S}$. We can also rule out $\frac{k}{4}$ (27 divisors: all but 8, 16, and 24), $\frac{k}{9}$ (27 divisors), $\frac{k}{25}$ (24 divisors), and all other numbers, thus the answer is $21 + 2 = 23$.

Problem 7. Let A and B be digits from the set $\{0, 1, 2, \ldots, 9\}$. Let r be the two-digit integer $\underline{A}\ \underline{B}$ and let s be the two-digit integer $\underline{B}\ \underline{A}$, so that r and s are members of the set $\{00, 01, \ldots, 99\}$. Compute the number of ordered pairs (A, B) such that $|r - s| = k^2$ for some integer k.

Solution 7. Because $|(10A+B) - (10B+A)| = 9|A - B| = k^2$, it follows that $|A - B|$ is a perfect square.

$|A - B| = 0$ yields 10 pairs of integers: $(A, B) = (0, 0), (1, 1), \ldots, (9, 9)$.
$|A - B| = 1$ yields 18 pairs: the nine $(A, B) = (0, 1), (1, 2), \ldots, (8, 9)$, and their reverses.
$|A - B| = 4$ yields 12 pairs: the six $(A, B) = (0, 4), (1, 5), \ldots, (5, 9)$, and their reverses.
$|A - B| = 9$ yields 2 pairs: $(A, B) = (0, 9)$ and its reverse.

Thus the total number of possible ordered pairs (A, B) is $10 + 18 + 12 + 2 = 42$.

Problem 8. For $k \geq 3$, we define an ordered k-tuple of real numbers $(x_1, x_2, \ldots, x_k)$ to be *special* if, for every i such that $1 \leq i \leq k$, the product $x_1 \cdot x_2 \cdot \ldots \cdot x_k = x_i^2$.Compute the smallest value of k such that there are at least 2009 distinct special k-tuples.

Solution 8. The given conditions imply k equations. By taking the product of these k equations, we have $(x_1 x_2 \ldots x_k)^{k-1} = x_1 x_2 \ldots x_k$. Thus it follows that either $x_1 x_2 \ldots x_k = 0$ or $x_1 x_2 \ldots x_k = \pm 1$. If $x_1 x_2 \ldots x_k = 0$, then some $x_j = 0$, and by plugging this into each of the equations, it follows that all of the x_i's are equal to 0. Note that we cannot have $x_1 x_2 \ldots x_k = -1$, because the left hand side equals $x_1(x_2 \ldots x_k) = x_1^2$, which can't be negative, because the x_i's are all given as real. Thus $x_1x_2\dots x_k = 1$, and it follows that each x_i is equal to either 1 or -1. Because the product of the x_i's is 1, there must be an even number of -1's. Furthermore, by picking any even number of the x_i's to be -1, it can be readily verified that the ordered k-tuple $(x_1, x_2, \dots, x_k)$ is special. Thus there are

$$\binom{k}{0} + \binom{k}{2} + \binom{k}{4} + \dots + \binom{k}{2\lfloor k/2 \rfloor}$$

special non-zero k-tuples. By considering the binomial expansion of $(1+1)^k + (1-1)^k$, it is clear that the above sum of binomial coefficients equals 2^{k-1}. Thus there are a total of $2^{k-1}+1$ special k-tuples. Because $2^{10} = 1024$ and $2^{11} = 2048$, the inequality $2^{k-1} + 1 \geq 2009$ is first satisfied when $k = 12$.

Alternate Solution: Use a recursive approach. Let S_k denote the number of special non-zero k-tuples. From the analysis in the above solution, each x_i must be either 1 or -1. It can easily be verified that $S_3 = 4$. For $k > 3$, suppose that $x_k = 1$ for a given special k-tuple. Then the k equations that follow are precisely the equation $x_1x_2\dots x_{k-1} = 1$ and the $k-1$ equations that follow for the special $(k-1)$-tuple $(x_1, x_2, \dots, x_{k-1})$. Because $x_1x_2\dots x_{k-1} = 1$ is consistent for a special $(k-1)$-tuple, and because this equation imposes no further restrictions, we conclude that there are S_{k-1} special k-tuples in which $x_k = 1$.

If, on the other hand, $x_k = -1$ for a given special k-tuple, then consider the k equations that result, and make the substitution $x_1 = -y_1$. Then the k resulting equations are precisely the same as the k equations obtained in the case where $x_k = 1$, except that x_1 is replaced by y_1. Thus $(x_1, x_2, \dots, x_{k-1}, -1)$ is special if and only if $(y_1, x_2, \dots, x_{k-1})$ is special, and thus there are S_{k-1} special k-tuples in which $x_k = -1$.

Thus the recursion becomes $S_k = 2S_{k-1}$, and because $S_3 = 4$, it follows that $S_k = 2^{k-1}$.

Problem 9. A cylinder with radius r and height h has volume 1 and total surface area 12. Compute $\frac{1}{r} + \frac{1}{h}$.

Solution 9. Since $\pi r^2 h = 1$, we have $h = \frac{1}{\pi r^2}$ and $\pi r^2 = \frac{1}{h}$. Consequently,

$$2\pi rh + 2\pi r^2 = 12 \Rightarrow (2\pi r)\left(\frac{1}{\pi r^2}\right) + 2\left(\frac{1}{h}\right) = 12 \Rightarrow \frac{2}{r} + \frac{2}{h} = 12 \Rightarrow \frac{1}{r} + \frac{1}{h} = 6.$$

Alternate Solution: The total surface area is $2\pi rh + 2\pi r^2 = 12$ and the volume is $\pi r^2 h = 1$. Dividing, we obtain $\frac{12}{1} = \frac{2\pi rh + 2\pi r^2}{\pi r^2 h} = \frac{2}{r} + \frac{2}{h}$, thus $\frac{1}{r} + \frac{1}{h} = \frac{12}{2} = 6$.

Problem 10. If $6\tan^{-1} x + 4\tan^{-1}(3x) = \pi$, compute x^2.

Solution 10. Let $z = 1 + xi$ and $w = 1 + 3xi$, where $i = \sqrt{-1}$. Then $\tan^{-1} x = \arg z$ and $\tan^{-1}(3x) = \arg w$, where $\arg z$ gives the measure of the angle in standard position whose terminal side passes through z. By DeMoivre's theorem, $6\tan^{-1} x = \arg(z^6)$ and $4\tan^{-1}(3x) = \arg(w^6)$. Therefore the equation $6\tan^{-1} x + 4\tan^{-1}(3x) = \pi$ is equivalent to $z^6 \cdot w^4 = a$, where a is a real number (and, in fact, $a < 0$). To simplify somewhat, we can take the square root of both sides, and get $z^3 \cdot w^2 = 0 + bi$, where b is a real number. Then $(1 + xi)^3(1 + 3xi)^2 = 0 + bi$. Expanding each binomial and collecting real and imaginary terms in each factor yields $((1-3x^2) + (3x - x^3)i)((1 - 9x^2) + 6xi) = 0 + bi$. In order that the real part of the product be 0, we have $(1 - 3x^2)(1 - 9x^2) - (3x - x^3)(6x) = 0$. This equation simplifies to $1 - 30x^2 + 33x^4 = 0$, yielding $x^2 = \frac{15 \pm 8\sqrt{3}}{33}$. Notice that $\frac{15 \pm 8\sqrt{3}}{33} \approx 1$, which would mean that $x \approx 1$, and so $\tan^{-1}(x) \approx \frac{\pi}{4}$, which is too large, since $6 \cdot \frac{\pi}{4} > \pi$. (It can be verified that this value for x yields a value of 3π for the left side of the equation.) Therefore we are left with $x^2 = \frac{15 - 8\sqrt{3}}{33}$.

To verify that this answer is reasonable, consider that $\sqrt{3} \approx 1.73$, so that $15 - 8\sqrt{3} \approx 1.16$, and so $x^2 \approx \frac{7}{200} = 0.035$. Then x itself is a little less than 0.2, and so $\tan^{-1} x \approx \frac{\pi}{15}$. Similarly, $3x$ is about 0.6, so $\tan^{-1}(3x)$ is about $\frac{\pi}{6}$. $6 \cdot \frac{\pi}{15} + 4 \cdot \frac{\pi}{6}$ is reasonably close to π.

Alternate Solution: Recall that $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$, thus $\tan(2a) = \frac{2\tan a}{1-\tan^2 a}$ and

$$\tan(3a) = \tan(2a+a) = \frac{\frac{2\tan a}{1-\tan^2 a} + \tan a}{1 - \frac{2\tan a}{1-\tan^2 a}\cdot \tan a} = \frac{2\tan a + \tan a - \tan^3 a}{1 - \tan^2 a - 2\tan^2 a} = \frac{3\tan a - \tan^3 a}{1 - 3\tan^2 a}.$$

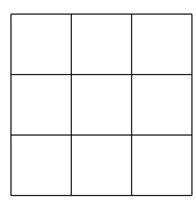
Back to the problem at hand, divide both sides by 2 to obtain $3\tan^{-1} x + 2\tan^{-1}(3x) = \frac{\pi}{2}$. Taking the tangent of the left side yields $\frac{\tan(3\tan^{-1} x) + \tan(2\tan^{-1}(3x))}{1 - \tan(3\tan^{-1} x)\tan(2\tan^{-1}(3x))}$. We know that the denominator must be 0 since $\tan\frac{\pi}{2}$ is undefined, thus $1 = \tan(3\tan^{-1} x)\tan(2\tan^{-1}(3x)) = \frac{3x-x^3}{1-3x^2}\cdot\frac{2\cdot 3x}{1-(3x)^2}$ and hence $(1-3x^2)(1-9x^2) = (3x-x^3)(6x)$. Simplifying yields $33x^4 - 30x^2 + 1 = 0$, and applying the quadratic formula gives $x^2 = \frac{15\pm 8\sqrt{3}}{33}$. The "+" solution is extraneous: as noted in the previous solution, $x = \frac{15+8\sqrt{3}}{33}$ yields a value of 3π for the left side of the equation), so we are left with $x^2 = \frac{15-8\sqrt{3}}{33}$.

4 Team Problems

Problem 1. Let N be a six-digit number formed by an arrangement of the digits $1, 2, 3, 3, 4, 5$. Compute the smallest value of N that is divisible by 264.

Problem 2. In triangle ABC, $AB = 4$, $BC = 6$, and $AC = 8$. Squares $ABQR$ and $BCST$ are drawn external to and lie in the same plane as $\triangle ABC$. Compute QT.

Problem 3. The numbers $1, 2, \ldots, 8$ are placed in the 3×3 grid below, leaving exactly one blank square. Such a placement is called *okay* if in every pair of adjacent squares, either one square is blank or the difference between the two numbers is at most 2 (two squares are considered adjacent if they share a common side). If reflections, rotations, etc. of placements are considered distinct, compute the number of distinct okay placements.



Problem 4. An ellipse in the first quadrant is tangent to both the x-axis and y-axis. One focus is at $(3, 7)$, and the other focus is at $(d, 7)$. Compute d.

Problem 5. Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a regular octagon. Let $\mathbf{u}$ be the vector from A_1 to A_2 and let $\mathbf{v}$ be the vector from A_1 to A_8. The vector from A_1 to A_4 can be written as $a\mathbf{u} + b\mathbf{v}$ for a unique ordered pair of real numbers (a, b). Compute (a, b).

Problem 6. Compute the integer n such that $2009 < n < 3009$ and the sum of the odd positive divisors of n is 1024.

Problem 7. Points A, R, M, and L are consecutively the midpoints of the sides of a square whose area is 650. The coordinates of point A are $(11, 5)$. If points R, M, and L are all lattice points, and R is in Quadrant I, compute the number of possible ordered pairs (x, y) of coordinates for point R.

Problem 8. The *taxicab distance* between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|.$$

The region $\mathcal{R}$ is obtained by taking the cube $\{(x, y, z) : 0 \leq x, y, z \leq 1\}$ and removing every point whose taxicab distance to any vertex of the cube is less than $\frac{3}{5}$. Compute the volume of $\mathcal{R}$.

Problem 9. Let a and b be real numbers such that

$$a^3 - 15a^2 + 20a - 50 = 0 \quad \text{and} \quad 8b^3 - 60b^2 - 290b + 2575 = 0.$$

Compute $a + b$.

Problem 10. For a positive integer n, define $s(n)$ to be the sum of n and its digits. For example, $s(2009) = 2009 + 2 + 0 + 0 + 9 = 2020$. Compute the number of elements in the set $\{s(0), s(1), s(2), \ldots, s(9999)\}$.

5 Team Answers

Answer 1. 135432

Answer 2. $2\sqrt{10}$

Answer 3. 32

Answer 4. $\frac{49}{3}$

Answer 5. $\left(2+\sqrt{2}, 1+\sqrt{2}\right)$

Answer 6. 2604

Answer 7. 10

Answer 8. $\frac{179}{250}$

Answer 9. $\frac{15}{2}$

Answer 10. 9046

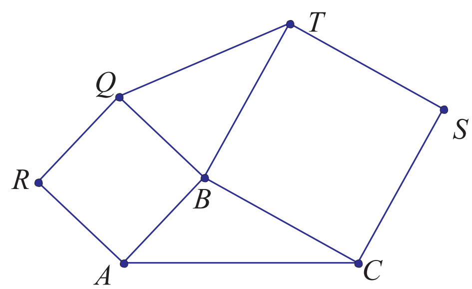
6 Team Solutions

Problem 1. Let N be a six-digit number formed by an arrangement of the digits $1, 2, 3, 3, 4, 5$. Compute the smallest value of N that is divisible by 264.

Solution 1. $264 = 3 \cdot 8 \cdot 11$, so we will need to address all these factors. Because the sum of the digits is 18, it follows that 3 divides N, regardless of how we order the digits of N. In order for 8 to divide N, we need N to end in $\underline{O}12, \underline{O}52, \underline{E}32$, or $\underline{E}24$, where O and E denote odd and even digits. Now write $N = \underline{U}\,\underline{V}\,\underline{W}\,\underline{X}\,\underline{Y}\,\underline{Z}$. Note that N is divisible by 11 if and only if $(U + W + Y) - (V + X + Z)$ is divisible by 11. Because the sum of the three largest digits is only 12, we must have $U + W + Y = V + X + Z = 9$.

Because Z must be even, this implies that V, X, Z are $2, 3, 4$ (in some order). This means $Y \neq 2$, and so we must have $Z \neq 4 \Rightarrow Z = 2$. Of the three remaining possibilities, $\underline{E}32$ gives the smallest solution, 135432.

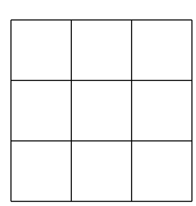
Problem 2. In triangle ABC, $AB = 4$, $BC = 6$, and $AC = 8$. Squares $ABQR$ and $BCST$ are drawn external to and lie in the same plane as $\triangle ABC$. Compute QT.



Solution 2. Set $m\angle ABC = x$ and $m\angle TBQ = y$. Then $x + y = 180^\circ$ and so $\cos x + \cos y = 0$. Applying the Law of Cosines to triangles ABC and TBQ gives $AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cos x$ and $QT^2 = BT^2 + BQ^2 - 2BT \cdot BQ \cos y$, which, after substituting values, become $8^2 = 4^2 + 6^2 - 48 \cos x$ and $QT^2 = 4^2 + 6^2 - 48 \cos y$. Adding the last two equations yields $QT^2 + 8^2 = 2(4^2 + 6^2)$ or $QT = 2\sqrt{10}$.

Remark: This problem is closely related to the fact that in a parallelogram, the sum of the squares of the lengths of its diagonals is the equal to the sum of the squares of the lengths of its sides.

Problem 3. The numbers $1, 2, \ldots, 8$ are placed in the 3×3 grid below, leaving exactly one blank square. Such a placement is called *okay* if in every pair of adjacent squares, either one square is blank or the difference between the two numbers is at most 2 (two squares are considered adjacent if they share a common side). If reflections, rotations, etc. of placements are considered distinct, compute the number of distinct okay placements.



Solution 3. We say that two numbers are *neighbors* if they occupy adjacent squares, and that a is a *friend* of b if $0 < |a-b| \le 2$. Using this vocabulary, the problem's condition is that every pair of neighbors must be friends of each other. Each of the numbers 1 and 8 has two friends, and each number has at most four friends.

If there is no number written in the center square, then we must have one of the cycles in the figures below. For each cycle, there are 8 rotations. Thus there are 16 possible configurations with no number written in the center square.

2	1	3
4	—	5
6	8	7

3	1	2
5	—	4
7	8	6

Now assume that the center square contains the number n. Because n has at least three neighbors, $n \neq 1$ and $n \neq 8$. First we show that 1 must be in a corner. If 1 is a neighbor of n, then one of the corners neighboring 1 must be empty, because 1 has only two friends (2 and 3). If c is in the other corner neighboring 1, then $\{n, c\} = \{2, 3\}$. But then n must have three more friends (n_1, n_2, n_3) other than 1 and c, for a total of five friends, which is impossible, as illustrated below. Therefore 1 must be in a corner.

—	1	c
n_1	n	n_2
	n_3	

Now we show that 1 can only have one neighbor, i.e., one of the squares adjacent to 1 is empty. If 1 has two neighbors, then we have, up to a reflection and a rotation, the configuration shown below. Because 2 has only one more friend, the corner next to 2 is empty and $n = 4$. Consequently, $m_1 = 5$ (refer to the figure below). Then 4 has one friend (namely 6) left with two neighbors m_2 and m_3, which is impossible. Thus 1 must have exactly one neighbor. An analogous argument shows that 8 must also be at a corner with exactly one neighbor.

1	2	—
3	n	m_3
m_1	m_2	

Therefore, 8 and 1 must be in non-opposite corners, with the blank square between them. Thus, up to reflections and rotations, the only possible configuration is the one shown at left below.

1	—	8
m		

1	—	8
2/3	4/5	6/7
3/2	5/4	7/6

There are two possible values for m, namely 2 and 3. For each of the cases $m = 2$ and $m = 3$, the rest of the configuration is uniquely determined, as illustrated in the figure above right. We summarize our process: there are four corner positions for 1; two (non-opposite) corner positions for 8 (after 1 is placed); and two choices for the number in the square neighboring 1 but not neighboring 8. This leads to $4 \cdot 2 \cdot 2 = 16$ distinct configurations with a number written in the center square.

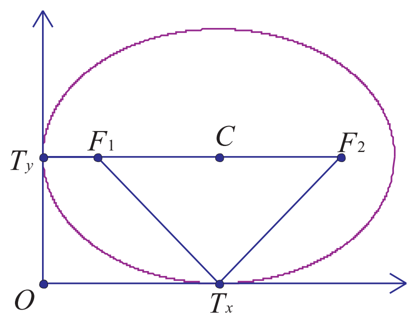
Therefore, there are 16 configurations in which the center square is blank and 16 configurations with a number in the center square, for a total of 32 distinct configurations.

Problem 4. An ellipse in the first quadrant is tangent to both the x-axis and y-axis. One focus is at $(3, 7)$, and the other focus is at $(d, 7)$. Compute d.

Solution 4. See the diagram below. The center of the ellipse is $C = \left(\frac{d+3}{2}, 7\right)$. The major axis of the ellipse is the line $y = 7$, and the minor axis is the line $x = \frac{d+3}{2}$. The ellipse is tangent to the coordinate axes at $T_x = \left(\frac{d+3}{2}, 0\right)$ and $T_y = (0, 7)$. Let $F_1 = (3, 7)$ and $F_2 = (d, 7)$. Using the locus definition of an ellipse, we have $F_1T_x + F_2T_x = F_1T_y + F_2T_y$; that is,

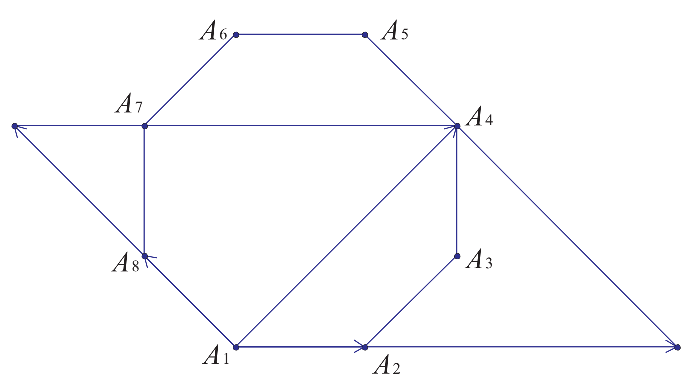
$$2\sqrt{\left(\frac{d-3}{2}\right)^2 + 7^2} = d + 3 \quad \text{or} \quad \sqrt{(d-3)^2 + 14^2} = d + 3.$$

Squaring both sides of the last equation gives $d^2 - 6d + 205 = d^2 + 6d + 9$ or $196 = 12d$, so $d = \frac{49}{3}$.



Problem 5. Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a regular octagon. Let $\mathbf{u}$ be the vector from A_1 to A_2 and let $\mathbf{v}$ be the vector from A_1 to A_8. The vector from A_1 to A_4 can be written as $a\mathbf{u} + b\mathbf{v}$ for a unique ordered pair of real numbers (a, b). Compute (a, b).

Solution 5. We can scale the octagon so that $A_1A_2 = \sqrt{2}$. Because the exterior angle of the octagon is $45°$, we can place the octagon in the coordinate plane with A_1 being the origin, $A_2 = (\sqrt{2}, 0)$, and $A_8 = (1, 1)$.



Then $A_3 = (1 + \sqrt{2}, 1)$ and $A_4 = (1 + \sqrt{2}, 1 + \sqrt{2})$. It follows that $\mathbf{u} = \left\langle \sqrt{2}, 0 \right\rangle$, $\mathbf{v} = \langle -1, 1 \rangle$, and

$$\overrightarrow{A_1A_4} = \left\langle 1 + \sqrt{2}, 1 + \sqrt{2} \right\rangle = a\left\langle \sqrt{2}, 0 \right\rangle + b\left\langle -1, 1 \right\rangle = \left\langle a\sqrt{2} - b, b \right\rangle.$$

Thus $b = \sqrt{2}+1$ and $a\sqrt{2} - b = \sqrt{2}+1$, or $a = 2+\sqrt{2}$.

Alternate Solution: Extend $\overline{A_1A_2}$ and $\overline{A_5A_4}$ to meet at point Q; let P be the intersection of $\overline{A_1Q}$ and $\overleftrightarrow{A_6A_3}$. Then $A_1A_2 = \|\mathbf{u}\|$, $A_2P = \|\mathbf{u}\|\sqrt{2}$, and $PQ = \|\mathbf{u}\|$, so $A_1Q = \left(2+\sqrt{2}\right)\|\mathbf{u}\|$. Because A_1QA_4 is a $45^\circ - 45^\circ - 90^\circ$ right triangle, $A_4Q = \frac{A_1Q}{\sqrt{2}} = \left(\sqrt{2}+1\right)\|\mathbf{u}\|$. Thus $\overrightarrow{A_1A_4} = \overrightarrow{A_1Q} + \overrightarrow{QA_4}$, and because $\|\mathbf{u}\| = \|\mathbf{v}\|$, we have $(a,b) = (2+\sqrt{2}, \sqrt{2}+1)$.

Problem 6. Compute the integer n such that $2009 < n < 3009$ and that the sum of the odd positive divisors of n is 1024.

Solution 6. Suppose that $n = 2^k p_1^{a_1} \cdots p_r^{a_r}$, where the p_i are distinct odd primes, k is a nonnegative integer, and $a_1, \dots, a_r$ are positive integers. Then the sum of the odd positive divisors of n is equal to

$$\prod_{i=1}^{r}(1+p_i+\cdots+p_i^{a_i}) = \prod_{i=1}^{r}\frac{p_i^{a_i+1}-1}{p_i-1} = 1024 = 2^{10}.$$

Note that $1+p_i+\cdots+p_i^{a_i}$ is the sum of a_i+1 odd numbers. Because the product of those sums is a power of two, each sum must be even (in fact, a power of 2). Thus, each a_i must be odd.

Because $1+11+11^2+11^3 > 1024$, if $p_i \geq 11$, then $a_i = 1$ and $1+p_i$ must be a power of 2 that is no greater than 1024. The possible values of p_i, with $p_i \geq 11$, are 31 and 127 (as 5 divides 255, 7 divides 511, and 3 divides 1023).

If $p_1 < 11$, then p_i can be 3, 5, 7. It is routine to check that $a_i = 1$ and $p_i = 3$ or 7.

Thus $a_i = 1$ for all i, and the possible values of p_i are $3, 7, 31, 127$. The only combinations of these primes that yield 1024 are $(1+3)\cdot(1+7)\cdot(1+31)$ (with $n = 2^k \cdot 3 \cdot 7 \cdot 31 = 651 \cdot 2^k$) and $(1+7)\cdot(1+127)$ (with $n = 7 \cdot 127 = 889 \cdot 2^k$). Thus $n = 651 \cdot 2^2 = 2604$ is the unique value of n satisfying the conditions of the problem.

Problem 7. Points A, R, M, and L are consecutively the midpoints of the sides of a square whose area is 650. The coordinates of point A are $(11, 5)$. If points R, M, and L are all lattice points, and R is in Quadrant I, compute all possible ordered pairs (x, y) of coordinates for point R.

Solution 7. Write $x = 11 + c$ and $y = 5 + d$. Then $AR^2 = c^2 + d^2 = \frac{1}{2} \cdot 650 = 325$. Note that $325 = 18^2 + 1^2 = 17^2 + 6^2 = 15^2 + 10^2$. Temporarily restricting ourselves to the case where c and d are both positive, there are three classes of solutions: $\{c, d\} = \{18, 1\}$, $\{c, d\} = \{17, 6\}$, or $\{c, d\} = \{15, 10\}$. In fact, c and d can be negative, so long as those values do not cause x or y to be negative. So there are ten solutions:

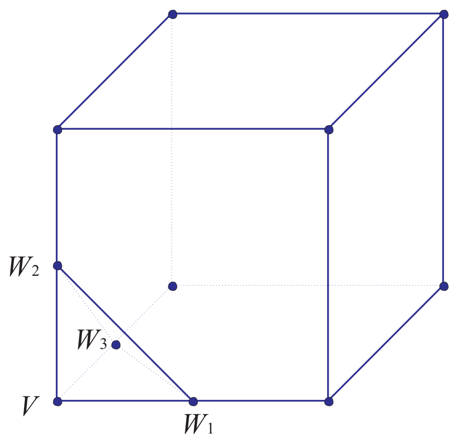
(c, d)	(x, y)
$(18, 1)$	$(29, 6)$
$(18, -1)$	$(29, 4)$
$(1, 18)$	$(12, 23)$
$(-1, 18)$	$(10, 23)$
$(17, 6)$	$(28, 11)$
$(6, 17)$	$(17, 22)$
$(-6, 17)$	$(5, 22)$
$(15, 10$	$(26, 15)$
$(10, 15)$	$(21, 20)$
$(-10, 15)$	$(1, 20)$

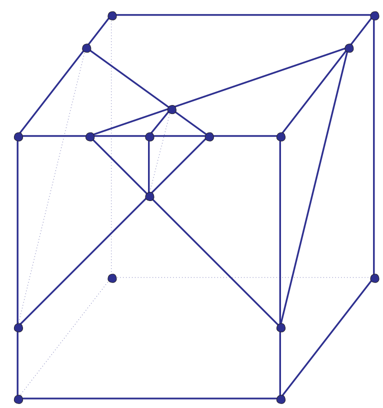
Problem 8. The *taxicab distance* between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|.$$

The region $\mathcal{R}$ is obtained by taking the cube $\{(x, y, z) : 0 \leq x, y, z \leq 1\}$ and removing every point whose taxicab distance to any vertex of the cube is less than $\frac{3}{5}$. Compute the volume of $\mathcal{R}$.

Solution 8. For a fixed vertex V on the cube, the locus of points on or inside the cube that are at most $\frac{3}{5}$ away from V form a *corner* at V (that is, the right pyramid $VW_1W_2W_3$ in the figure shown at left below, with equilateral triangular base $W_1W_2W_3$ and three isosceles right triangular lateral faces $VW_1W_2, VW_2W_3, VW_3W_1$). Thus $\mathcal{R}$ is formed by removing eight such congruent corners from the cube. However, each two neighboring corners share a common region along their shared edge. This common region is the union of two smaller right pyramids, each similar to the original corners. (See the figure shown at right below.)





We compute the volume of $\mathcal{R}$ as

$$1 - 8 \cdot \frac{1}{6}\left(\frac{3}{5}\right)^3 + 12 \cdot 2 \cdot \frac{1}{6}\left(\frac{1}{10}\right)^3 = \frac{179}{250}.$$

Problem 9. Let a and b be real numbers such that

$$a^3 - 15a^2 + 20a - 50 = 0 \quad \text{and} \quad 8b^3 - 60b^2 - 290b + 2575 = 0.$$

Compute $a + b$.

Solution 9. Each cubic expression can be depressed—that is, the quadratic term can be eliminated—by substituting as follows. Because $(a - p)^3 = a^3 - 3a^2p + 3ap^2 - p^3$, setting $p = -\frac{(-15)}{3} = 5$ and substituting $c + p = a$ transforms the expression $a^3 - 15a^2 + 20a - 50$ into the equivalent expression $(c+5)^3 - 15(c+5)^2 + 20(c+5) - 50$, which simplifies to $c^3 - 55c - 200$. Similarly, the substitution $d = b - \frac{5}{2}$ yields the equation $d^3 - 55d = -200$. [This procedure, which is analogous to completing the square, is an essential step in the algebraic solution to the general cubic equation.]

Consider the function $f(x) = x^3 - 55x$. It has three zeros, namely, 0 and $\pm\sqrt{55}$. Therefore, it has a relative maximum and a relative minimum in the interval $[-\sqrt{55}, \sqrt{55}]$. Note that for $0 \leq x \leq 5.5$, $|f(x)| < |x^3| < 5.5^3 = 166.375$, and for $5.5 < x \leq \sqrt{55} < 8$, we have

$$|f(x)| = |x^3 - 55x| < x|x^2 - 55| < 8(55 - 5.5^2) = 198.$$

Because $f(x)$ is an odd function of x (its graph is symmetric about the origin), we conclude that for $-\sqrt{55} \leq x \leq \sqrt{55}$, $|f(x)| < 198$. Therefore, for constant m with $|m| > 198$, there is a unique real number x_0 such that $f(x_0) = m$.

In particular, since $200 > 198$, the values of c and d are uniquely determined. Because $f(x)$ is odd, we conclude that $c = -d$, or $a + b = \frac{15}{2}$.

Alternate Solution: Set $a = x - b$ and substitute into the first equation. We get

$$\begin{aligned} (x-b)^3 - 15(x-b)^2 + 20(x-b) - 50 &= 0 \\ -b^3 + b^2(3x-15) + b(-3x^2+30x-20) + (x^3-15x^2+20x-50) &= 0 \\ 8b^3 + b^2(-24x+120) + b(24x^2-240x+160) - 8(x^3-15x^2+20x-50) &= 0. \end{aligned}$$

If we equate coefficients, we see that

$$\begin{aligned} -24x + 120 &= -60 \\ 24x^2 - 240x + 160 &= -290 \\ -8(x^3 - 15x^2 + 20x - 50) &= 2575 \end{aligned}$$

are all satisfied by $x = \frac{15}{2}$. This means that any real solution b to the second equation yields a real solution of $\frac{15}{2} - b$ to the first equation. We can follow the reasoning of the previous solution to establish the existence of exactly one real solution to the second cubic equation. Thus a and b are unique, and their sum is $(\frac{15}{2} - b) + b = \frac{15}{2}$.

Problem 10. For a positive integer n, define $s(n)$ to be the sum of n and its digits. For example, $s(2009) = 2009 + 2 + 0 + 0 + 9 = 2020$. Compute the number of elements in the set $\{s(0), s(1), s(2), \ldots, s(9999)\}$.

Solution 10. If $s(10x) = a$, then the values of s over $\{10x+0, 10x+1, \ldots, 10x+9\}$ are a, $a+2$, $a+4, \ldots, a+18$. Furthermore, if x is not a multiple of 10, then $s(10(x+1)) = a + 11$. This indicates that the values of s "interweave" somewhat from one group of 10 to the next: the sets alternate between even and odd. Because the s-values for starting blocks of ten differ by 11, consecutive blocks of the same parity differ by 22, so the values of s do not overlap. That is, s takes on 100 distinct values over any range of the form $\{100y+0, 100y+1, \ldots, 100y+99\}$.

First determine how many values are repeated between consecutive hundreds. Let y be an integer that is not a multiple of 10. Then the largest value for $s(100y+k)$ $(0 \leq k \leq 99)$ is $100y + (s(y) - y) + 99 + s(99) = 100y + s(y) - y + 117$, whereas the smallest value in the next group of 100 is for

$$\begin{aligned} s(100(y+1)) &= 100(y+1) + (s(y+1) - (y+1)) = 100y + (s(y) + 2) - (y+1) + 100 \\ &= 100y + s(y) - y + 101. \end{aligned}$$

This result implies that the values for $s(100y+91)$ through $s(100y+99)$ match the values of $s(100y+100)$ through $s(100y+108)$. So there are 9 repeated values.

Now determine how many values are repeated between consecutive thousands. Let z be a digit, and consider $s(1000z + 999)$ versus $s(1000(z+1))$. The first value equals

$$1000z + (s(z) - z) + 999 + s(999) = 1000z + z + 1026 = 1001z + 1026.$$

The latter value equals $1000(z+1) + (s(z+1) - (z+1)) = 1001(z+1) = 1001z + 1001$. These values differ by an odd number. We have overlap between the $982, 983, \ldots, 989$ terms and the $000, 001, \ldots, 007$ terms. We also have overlap between the $992, 993, \ldots, 999$ terms and the $010, 011, \ldots, 017$ terms, for a total of 16 repeated values in all.

There are 90 instances in which we have 9 repeated terms, and 9 instances in which we have 16 repeated terms, so there are a total of $10000 - 90 \cdot 9 - 9 \cdot 16 = 9046$ unique values.

7 Power Question: Sign on the Label

Instructions: The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute", you need not justify your answer. If a problem says "determine" or "find", then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items but not vice-versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the Team name) on the cover sheet used as the first page of the papers submitted. Do not identify the Team in any other way.

An ***n*-label** is a permutation of the numbers 1 through n. For example, $J = 35214$ is a 5-label and $K = 132$ is a 3-label. For a fixed positive integer p, where $p \leq n$, consider consecutive blocks of p numbers in an n-label. For example, when $p = 3$ and $L = 263415$, the blocks are $263, 634, 341$, and 415. We can associate to each of these blocks a p-label that corresponds to the relative order of the numbers in that block. For $L = 263415$, we get the following:

$$\underline{263}415 \rightarrow 132; \quad 2\underline{634}15 \rightarrow 312; \quad 26\underline{341}5 \rightarrow 231; \quad 263\underline{415} \rightarrow 213.$$

Moving from left to right in the n-label, there are $n-p+1$ such blocks, which means we obtain an $(n-p+1)$-tuple of p-labels. For $L = 263415$, we get the 4-tuple $(132, 312, 231, 213)$. We will call this $(n-p+1)$-tuple the ***p*-signature** of L (or **signature**, if p is clear from the context) and denote it by $S_p[L]$; the p-labels in the signature are called **windows**. For $L = 263415$, the windows are $132, 312, 231$, and 213, and we write

$$S_3[263415] = (132, 312, 231, 213).$$

More generally, we will call *any* $(n-p+1)$-tuple of p-labels a p-signature, even if we do not know of an n-label to which it corresponds (and even if no such label exists). A signature that occurs for exactly one n-label is called **unique**, and a signature that doesn't occur for any n-labels is called **impossible**. A **possible** signature is one that occurs for at least one n-label.

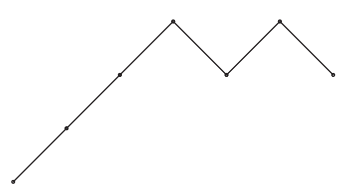
In this power question, you will be asked to analyze some of the properties of labels and signatures.

The Problems

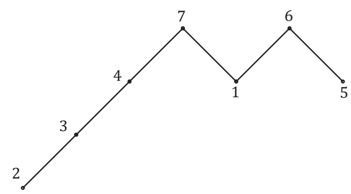
1. (a) Compute the 3-signature for 52341. [1]
   (b) Find another 5-label with the same 3-signature as in part (a). [2]
   (c) Compute two other 6-labels with the same 4-signature as 462135. [2]

2. (a) Explain why the label 1234 has a unique 3-signature. [1]
   (b) List three other 4-labels with unique 3-signatures. [1]
   (c) Explain why the 3-signature $(123, 321)$ is impossible. [1]
   (d) List three other impossible 3-signatures that have exactly two windows. [1]

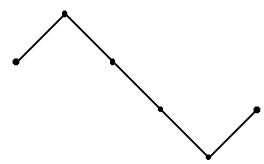
We can associate a **shape** to a given 2-signature: a diagram of up and down steps that indicates the relative order of adjacent numbers. For example, the following shape corresponds to the 2-signature $(12, 12, 12, 21, 12, 21)$:



A 7-label with this 2-signature corresponds to placing the numbers 1 through 7 at the nodes above so that numbers increase with each up step and decrease with each down step. The 7-label 2347165 is shown below:



3. Consider the shape below:



   (a) Find the 2-signature that corresponds to this shape. [2]

   (b) Compute two different 6-labels with the 2-signature you found in part (a). [3]

4. (a) List all 5-labels with 2-signature $(12, 12, 21, 21)$. [2]

   (b) Find a formula for the number of $(2n + 1)$-labels with the 2-signature [3]

$$(\underbrace{12, 12, \ldots, 12}_{n}, \underbrace{21, 21, \ldots, 21}_{n}).$$

5. (a) Compute the number of 5-labels with 2-signature $(12, 21, 12, 21)$. [2]

   (b) Determine the number of 9-labels with 2-signature

$$(12, 21, 12, 21, 12, 21, 12, 21).$$

   Justify your answer. [3]

6. (a) Determine whether the following signatures are possible or impossible:

      (i) $(123, 132, 213)$, [1]

      (ii) $(321, 312, 213)$. [1]

   (b) Notice that a $(p + 1)$-label has only two windows in its p-signature. For a given window ω_1, compute the number of windows ω_2 such that $S_p[L] = (\omega_1, \omega_2)$ for some $(p + 1)$-label L. [1]

   (c) Justify your answer from part (b). [2]

7. (a) For a general n, determine the number of distinct possible p-signatures. [4]

   (b) If a randomly chosen p-signature is 575 times more likely of being impossible than possible, determine p and n. [1]

8. (a) Show that $(312, 231, 312, 132)$ is not a unique 3-signature. [1]

   (b) Show that $(231, 213, 123, 132)$ is a unique 3-signature. [2]

   (c) Find two 5-labels with unique 2-signatures. [1]

(d) Find a 6-label with a unique 4-signature but which has the 3-signature from part (a). [2]

9. (a) For a general $n \geq 2$, compute all n-labels that have unique 2-signatures. [1]

(b) Determine whether or not $S_5[495138627]$ is unique. [1]

(c) Determine the smallest p for which the 20-label

$$L = 3, 11, 8, 4, 17, 7, 15, 19, 6, 2, 14, 1, 10, 16, 5, 12, 20, 9, 13, 18$$

has a unique p-signature. [3]

10. Show that for each $k \geq 2$, the number of unique 2^{k-1}-signatures on the set of 2^k-labels is at least 2^{2^k-3}. [5]

8 Power Solutions

1 (a) $(312, 123, 231)$

(b) There are three: $41352, 42351, 51342$.

(c) There are three: $362145, 452136, 352146$.

2 (a) We will prove this by contradiction. Suppose for some other 4-label L we have $S_3[L] = S_3[1234] = (123, 123)$. Write out L as a_1, a_2, a_3, a_4. From the first window of $S_3[L]$, we have $a_1 < a_2 < a_3$. From the second window, we have $a_2 < a_3 < a_4$. Connecting these inequalities gives $a_1 < a_2 < a_3 < a_4$, which forces $L = 1234$, a contradiction. Therefore, the 3-signature above is unique.

(b) There are 11 others (12 in all, if we include $S_3[1234]$):

$$\begin{array}{lll} S_3[1234] = (123, 123) & S_3[1243] = (123, 132) & S_3[1324] = (132, 213) \\ S_3[1423] = (132, 213) & S_3[2134] = (213, 123) & S_3[2314] = (231, 213) \\ S_3[3241] = (213, 231) & S_3[3421] = (231, 321) & S_3[4132] = (312, 132) \\ S_3[4231] = (312, 231) & S_3[4312] = (321, 312) & S_3[4321] = (321, 321) \end{array}$$

(c) If $S_3[a_1, a_2, a_3, a_4] = (123, 321)$, then the first window forces $a_2 < a_3$, whereas the second window forces $a_2 > a_3$. This is impossible, so the 3-signature $(123, 321)$ is impossible.

(d) There are 18 impossible 3-signatures with two windows. In nine of these, the first window indicates that $a_2 < a_3$ (an increase), but the second window indicates that $a_2 > a_3$ (a decrease). In the other nine, the end of the first window indicates a decrease (that is, $a_2 > a_3$), but the beginning of the second window indicates an increase ($a_2 < a_3$). In general, for a 3-signature to be possible, the end of the first window and beginning of the second window must be consistent, indicating either an increase or a decrease. The impossible 3-signatures are

$$\begin{array}{cccccc} (123, 321) & (123, 312) & (123, 213) & (132, 231) & (132, 132) & (132, 123) \\ (213, 321) & (213, 312) & (213, 213) & (231, 231) & (231, 132) & (231, 123) \\ (312, 321) & (312, 312) & (312, 213) & (321, 231) & (321, 132) & (321, 123) \end{array}$$

3. (a) The first pair indicates an increase; the next three are decreases, and the last pair is an increase. So the 2-signature is $(12, 21, 21, 21, 12)$.

(b) There are several:

564312	564213	563214	465312	465213	463215
365412	365214	364215	265413	265314	264315
			165413	165314	164315
		453216	354216	254316	154326

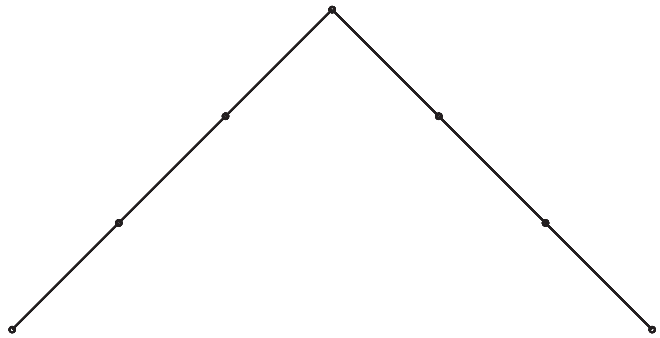
4 In part (a), we can count by brute force, or use the formula from part (b) (with indepedent proof).

(a) For the case of 5-labels, brute force counting is tractable.

$$\boxed{12543, 13542, 14532, 23541, 24531, 34521}.$$

(b) The answer is $\boxed{\binom{2n}{n}}$.

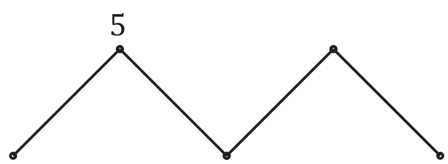
The shape of this signature is a wedge: n up steps followed by n down steps. The wedge for $n = 3$ is illustrated below:



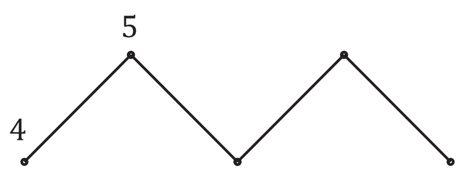
The largest number in the label, $2n+1$, must be placed at the peak in the center. If we choose the numbers to put in the first n spaces, then they must be placed in increasing order. Likewise, the remaining n numbers must be placed in decreasing order on the downward sloping piece of the shape. Thus there are exactly $\binom{2n}{n}$ such labels.

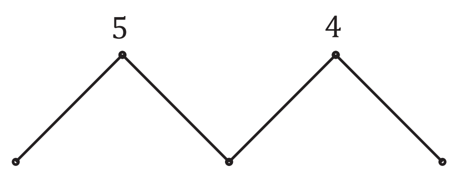
5 (a) The answer is $\boxed{16}$.

We have a shape with two peaks and a valley in the middle. The 5 must go on one of the two peaks, so we place it on the first peak. By the shape's symmetry, we will double our answer at the end to account for the 5-labels where the 5 is on the other peak.



The 4 can go to the left of the 5 or at the other peak. In the first case, shown below left, the 3 must go at the other peak and the 1 and 2 can go in either order. In the latter case, shown below right, the $1, 2$, and 3 can go in any of $3!$ arrangements.

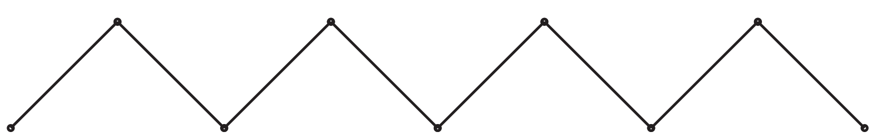




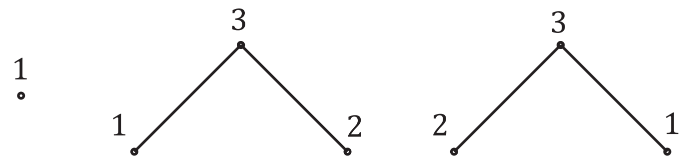
So there are $2! + 3! = 8$ possibilities. In all, there are 16 5-labels (including the ones where the 5 is at the other peak).

(b) The answer is $\boxed{7936}$.

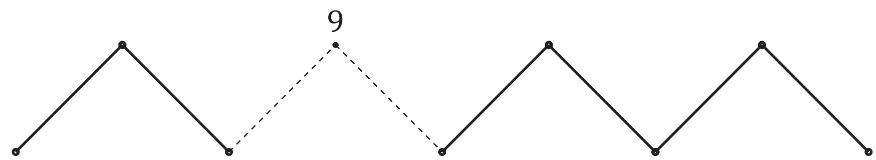
The shape of this 2-signature has four peaks and three intermediate valleys:



We will solve this problem by building up from smaller examples. Let f_n equal the number of $(2n+1)$-labels whose 2-signature consists of n peaks and $n-1$ intermediate valleys. In part (b) we showed that $f_2 = 16$. In the case where we have one peak, $f_1 = 2$. For the trivial case (no peaks), we get $f_0 = 1$. These cases are shown below.



Suppose we know the peak on which the largest number, $2n+1$, is placed. Then that splits our picture into two shapes with fewer peaks. Once we choose which numbers from $1, 2, \ldots, 2n$ to place each shape, we can compute the number of arrangments of the numbers on each shape, and then take the product. For example, if we place the 9 at the second peak, as shown below, we get a 1-peak shape on the left and a 2-peak shape on the right.



For the above shape, there are $\binom{8}{3}$ ways to pick the three numbers to place on the lefthand side, $f_1 = 2$ ways to place them, and $f_2 = 16$ ways to place the remaining five numbers on the right.

This argument works for any $n > 1$, so we have shown the following:

$$f_n = \sum_{k=1}^{n} \binom{2n}{2k-1} f_{k-1} f_{n-k}.$$

So we have:

$$\begin{aligned}
f_1 &= \binom{2}{1} f_0^2 = 2 \\
f_2 &= \binom{4}{1} f_0 f_1 + \binom{4}{3} f_1 f_0 = 16 \\
f_3 &= \binom{6}{1} f_0 f_2 + \binom{6}{3} f_1^2 + \binom{6}{5} f_2 f_0 = 272 \\
f_4 &= \binom{8}{1} f_0 f_3 + \binom{8}{3} f_1 f_2 + \binom{8}{5} f_2 f_1 + \binom{8}{7} f_3 f_0 = \boxed{7936}.
\end{aligned}$$

6 (a) Signature (i) is possible, because it is the 3-signature of 12435.

Signature (ii) is impossible. Let a 5-label be a_1, a_2, a_3, a_4, a_5. The second window of (ii) implies $a_3 < a_4$, whereas the third window implies $a_3 > a_4$, a contradiction.

(b) There are $\boxed{p}$ such windows, regardless of ω_1.

(c) Because the windows ω_1 and ω_2 overlap for $p-1$ numbers in L, the first $p-1$ integers in ω_2 must be in the same relative order as the last $p-1$ integers in ω_1. Therefore, by choosing the last integer in ω_2 (there are p choices), the placement of the remaining $p-1$ integers is determined. We only need to show that there exists some L such that $S_p[L] = (\omega_1, \omega_2)$.

To do so, set $\omega_1 = w_1, w_2, \ldots, w_p$. Provisionally, let $L^{(k)} = w_1, w_2, \ldots, w_p, k + 0.5$ for $k = 0, 1, \ldots, p$. For example, if $p = 4$ and $\omega_1 = 3124$, then $L^{(2)} = 3, 1, 2, 4, 2.5$. First we show that $L^{(k)}$ has the required

p-signature; then we can renumber the entries in $L^{(k)}$ in consecutive order to make them all integers, so for example $3, 1, 2, 4, 2.5$ would become $4, 1, 2, 5, 3$.

Even though the last entry in $L^{(k)}$ is not an integer, we can still compute $S_p[L^{(k)}]$, since all we require is that the entries are all distinct. In each $S_p[L^{(k)}]$, the first window is ω_1. When we compare $L^{(k)}$ to $L^{(k+1)}$, the last integer in ω_2 can increase by at most 1, because the last integer in the label jumps over at most one integer in positions 2 through p (those jumps are in boldface):

$$
\begin{array}{lll}
L^{(0)} = 3, 1, 2, 4, 0.5 & S_4[L^{(0)}] = (3124, 234\underline{1}) & \\
L^{(1)} = 3, \mathbf{1}, 2, 4, 1.5 & S_4[L^{(1)}] = (3124, 134\underline{2}) & \text{[jumps over the 1]} \\
L^{(2)} = 3, 1, \mathbf{2}, 4, 2.5 & S_4[L^{(2)}] = (3124, 124\underline{3}) & \text{[jumps over the 2]} \\
L^{(3)} = 3, 1, 2, 4, 3.5 & S_4[L^{(3)}] = (3124, 124\underline{3}) & \text{[jumps over nothing]} \\
L^{(4)} = 3, 1, 2, \mathbf{4}, 4.5 & S_4[L^{(4)}] = (3124, 123\underline{4}) & \text{[jumps over the 4]}
\end{array}
$$

The final entry of ω_2 in $S_p[L^{(0)}]$ is 1, because 0.5 is smaller than all other entries in $L^{(0)}$. Likewise, the final entry of ω_2 in $S_p[L^{(p)}]$ is p. Since the final entry increases in increments of 0 or 1 (as underlined above), we must see all p possibilities for ω_2.

By replacing the numbers in each $L^{(k)}$ with the integers 1 through $p+1$ (in the same relative order as the numbers in $L^{(k)}$), we have found the $(p+1)$-labels that yield all p possibilities.

7 (a) The answer is $\boxed{p! \cdot p^{n-p}}$.

Call two consecutive windows in a p-signature **compatible** if the last $p-1$ numbers in the first label and the first $p-1$ numbers in the second label (their "overlap") describe the same ordering. For example, in the p-signature $(\ldots, 2143, 2431, \ldots)$, 2143 and 2431 are compatible. Notice that the last three digits of 2143 and the first three digits of 2431 can be described by the same 3-label, 132.

Theorem. A signature σ is possible if and only if every pair of consecutive windows is compatible.

Proof. ($\Rightarrow$) Consider a signature σ describing a p-label L. If some pair in σ is not compatible, then there is some string of $p-1$ numbers in our label L that has two different $(p-1)$-signatures. This is impossible, since the p-signature is well-defined.

($\Leftarrow$) Now suppose σ is a p-signature such that that every pair of consecutive windows is compatible. We need to show that there is at least one label L with $S_p[L] = \sigma$. We do so by induction on the number of windows in σ, using the results from 5(b).

Let $\sigma = \{\omega_1, \omega_2, \ldots, \omega_{k+1}\}$, and suppose $\omega_1 = a_1, a_2, \ldots, a_p$. Set $L_1 = \omega_1$.

Suppose that L_k is a $(p+k-1)$-label such that $S_p[L_k] = \{\omega_1, \ldots, \omega_k\}$. We will construct L_{k+1} for which $S_p[L_{k+1}] = \{\omega_1, \ldots, \omega_{k+1}\}$.

As in 5(b), denote by $L_k^{(j)}$ the label L_k with a $j + 0.5$ appended; we will eventually renumber the elements in the label to make them all integers. Appending $j + 0.5$ does not affect any of the non-terminal windows of $S_p[L_k]$, and as j varies from 0 to $p - k + 1$ the final window of $S_p[L_k^{(j)}]$ varies over each of the p windows compatible with ω_k. Since ω_{k+1} is compatible with ω_k, there exists some j for which $S_p[L_k^{(j)}] = \{\omega_1, \ldots, \omega_{k+1}\}$. Now we renumber as follows: set $L_{k+1} = S_{k+p}[L_k^{(j)}]$, which replaces $L_k^{(j)}$ with the integers 1 through $k + p$ and preserves the relative order of all integers in the label.

By continuing this process, we conclude that the n-label L_{n-p+1} has p-signature σ, so σ is possible. $\square$

To count the number of possible p-signatures, we choose the first window ($p!$ choices), then choose each of the remaining $n-p$ compatible windows (p choices each). In all, there are $p! \cdot p^{n-p}$ possible p-signatures.

(b) The answer is $\boxed{n = 7, p = 5}$.

Let P denote the probability that a randomly chosen p-signature is possible. We are given that $1-P = 575$, so $P = \frac{1}{576}$. We want to find p and n for which

$$\begin{aligned} \frac{p! \cdot p^{n-p}}{(p!)^{n-p+1}} &= \frac{1}{576} \\ \frac{p^{n-p}}{(p!)^{n-p}} &= \frac{1}{576} \\ ((p-1)!)^{n-p} &= 576. \end{aligned}$$

The only factorial that has 576 as an integer power is $4! = \sqrt{576}$. Thus $p = 5$ and $n - p = 2 \Rightarrow n = 7$.

8 (a) The p-signature is not unique because it equals both $S_3[625143]$ and $S_3[635142]$.

(b) Let $L = a_1, a_2, a_3, a_4, a_5, a_6$. We have $a_4 < a_6 < a_5$ (from window #4), $a_3 < a_1 < a_2$ (from window #1), and $a_2 < a_4$ (from window #2). Linking these inequalities, we get

$$a_3 < a_1 < a_2 < a_4 < a_6 < a_5 \quad \Rightarrow \quad L = 231456,$$

so $S_3[L]$ is unique.

(c) 12345 and 54321 are the only ones.

(d) $L = 645132$ will work. First, note that $S_3[645132] = (312, 231, 312, 132)$. Next, we need to show that $S_4[645132] = \{4231, 3412, 4132\}$ is unique. So let $L' = a_1, a_2, a_3, a_4, a_5, a_6$ be a 6-label such that $S_4[L'] = (4231, 3412, 4132)$.

We get $a_4 < a_6 < a_5$ (from window #3), $a_5 < a_2 < a_3$ (from window #2), and $a_3 < a_1$ (from window #1). Linking these inequalities, we get

$$a_4 < a_6 < a_5 < a_2 < a_3 < a_1 \quad \Rightarrow \quad L' = 645132,$$

so $L' = L$, which means $S_4[L]$ is unique.

9 (a) The n-labels with unique 2-signatures are $1, 2, \ldots, n$ and $n, n-1, \ldots, 1$, and their respective 2-signatures are $(12, 12, \ldots, 12)$ and $(21, 21, \ldots, 21)$.

Proof: (Not required for credit.) Let $L = a_1, a_2, \ldots, a_n$. The first signature above implies that $a_1 < a_2 < \cdots < a_n$, which forces $a_1 = 1, a_2 = 2$, and so on. Likewise, the second signature forces $a_1 = n, a_2 = n-1$, and so on.

To show that all other n-labels fail to have unique 2-signatures, we will show that in any other n-label L', there are two numbers k and $k+1$ that are not adjacent. By switching k and $k+1$ we get a label L'' for which $S_2[L'] = S_2[L'']$, since the differences between k and its neighbors in L' were at least 2 (and likewise for $k+1$).

To show that such a k and $k+1$ exist, we proceed by contradiction. Suppose that all such pairs are adjacent in L'. Then 1 and n must be at the ends of L' (or else some intermediate number k will fail to be adjacent to $k+1$ or $k-1$). But if $a_1 = 1$, then that forces $a_2 = 2$, $a_3 = 3, \ldots, a_n = n$. That is, we get the two labels we already covered. (We get the other label if $a_n = 1$.)

Therefore, none of the $n! - 2$ remaining labels has a unique 2-signature.

(b) $S_5[495138627]$ is unique.

Let $L = a_1, \ldots, a_9$ and suppose $S_5[L] = S_5[495138627] = (\omega_1, \ldots, \omega_5)$. Then we get the following inequalities:

$a_4 < a_8$	[from ω_4]	$a_8 < a_5$	[from ω_4]
$a_5 < a_1$	[from ω_1]	$a_1 < a_3$	[from ω_1]
$a_3 < a_7$	[from ω_3]	$a_7 < a_9$	[from ω_5]
$a_9 < a_6$	[from ω_5]	$a_6 < a_2$	[from ω_2]

Combining, we get $a_4 < a_8 < a_5 < a_1 < a_3 < a_7 < a_9 < a_6 < a_2$, which forces $a_4 = 1, a_8 = 2, \ldots, a_2 = 9$. So the label L is forced and $S_5[495138627]$ is therefore unique.

(c) The answer is $\boxed{p = 16}$. To show this fact we will need to extend the idea from part 8(b) about "linking" inequalities forced by the various windows:

Theorem: A p-signature for an n-label L is unique if and only if for every $k < n$, k and $k+1$ are in at least one window together. That is, the **distance** between them in the n-label is less than p.

Proof. Suppose that for some k, the distance between k and $k+1$ is p or greater. Then the label L' obtained by swapping k and $k+1$ has the same p-signature, because there are no numbers between k and $k+1$ in any window and because the two numbers never appear in the same window.

If the distance between all such pairs is less than k, we need to show that $S_p[L]$ is unique. For $i = 1, 2, \ldots, n$, let r_i denote the position where i appears in L. For example, if $L = 4123$, then $r_1 = 2, r_2 = 3, r_3 = 4$, and $r_4 = 1$.

Let $L = a_1, a_2, \ldots, a_n$. Since 1 and 2 are in some window together, $a_{r_1} < a_{r_2}$. Similarly, for any k, since k and $k+1$ are in some window together, $a_{r_k} < a_{r_{k+1}}$. We then get a linked inequality $a_{r_1} < a_{r_2} < \cdots < a_{r_n}$, which can only be satisfied if $a_{r_1} = 1, a_{r_2} = 2, \ldots, a_{r_n} = n$. Therefore, $S_p[L]$ is unique. $\square$

From the proof above, we know that the signature is unique if and only if every pair of consecutive integers coexists in at least one window. Therefore, we seek the largest distance between consecutive integers in L. That distance is 15 (from 8 to 9, and from 17 to 18). Thus the smallest p is $\boxed{16}$.

10 Let s_k denote the number of such unique signatures. We proceed by induction with base case $k = 2$. From 8(c), a 2-signature for a label L is unique if and only if consecutive numbers in L appear together in some window. Because $k = 2$, the consecutive numbers must be adjacent in the label. The 2^2-labels 1234 and 4321 satisfy this condition, [1] so their 2^1-signatures are unique. Thus we have shown that $s_2 \geq 2 = 2^{2^2-3}$, and the base case is established.

Now suppose $s_k \geq 2^{2^k-3}$ for some $k \geq 2$. Let L_k be a 2^k-label with a unique 2^{k-1}-signature. Write $L_k = (a_1, a_2, \ldots, a_{2^k})$. We will *expand* L_k to form a 2^{k+1} label by replacing each a_i above with the numbers $2a_i - 1$ and $2a_i$ (in some order). This process produces a valid 2^{k+1}-label, because the numbers produced

[1] The proof in 8(a) establishes that these are the only 2^2-labels with unique 2-signatures, thus giving $s_2 = 2$. That proof was not required for credit and is not needed here, since the inequality above is good enough for this problem.

are all the integers from 1 to 2^{k+1}. Furthermore, different L_k's will produce different labels: if the starting labels differ at place i, then the new labels will differ at places $2i-1$ and $2i$. Therefore, each starting label produces 2^{2^k} distinct 2^{k+1}-labels through this process. Summarizing, each valid 2^k-label can be expanded to produce 2^{2^k} distinct 2^{k+1}-labels, none of which could be obtained by expanding any other 2^k-label.

It remains to be shown that the new label has a unique 2^{k-1}-signature. Because L_k has a unique 2^{k-1}-signature, for all $i \leq 2^k - 1$, both i and $i+1$ appeared in some 2^{k-1}-window. Therefore, there were fewer than $2^{k-1} - 1$ numbers between i and $i+1$. When the label is expanded, $2i$ and $2i-1$ are adjacent, $2i+1$ and $2i+2$ are adjacent, and $2i$ and $2i+1$ are fewer than $2 \cdot (2^{k-1} - 1) + 2 = 2^k$ places apart. Thus, every pair of adjacent integers is within some 2^k-window.

Since each pair of consecutive integers in our new 2^{k+1}-label coexists in some 2^k-window for every possible such expansion of L_k, that means all 2^{2^k} ways of expanding L_k to a 2^{k+1}-label result in labels with unique 2^k-signatures. We then get

$$\begin{aligned} s_{k+1} &\geq 2^{2^k} \cdot s_k \\ &\geq 2^{2^k} \cdot 2^{2^k-3} \\ &= 2^{2^{k+1}-3}, \end{aligned}$$

which completes the induction.

9 Relay Problems

Relay 1-1 A rectangular box has dimensions $8 \times 10 \times 12$. Compute the fraction of the box's volume that is *not* within 1 unit of any of the box's faces.

Relay 1-2 Let $T = TNYWR$. Compute the largest real solution x to $(\log x)^2 - \log \sqrt{x} = T$.

Relay 1-3 Let $T = TNYWR$. Kay has $T + 1$ different colors of fingernail polish. Compute the number of ways that Kay can paint the five fingernails on her left hand by using at least three colors and such that no two consecutive fingernails have the same color.

Relay 2-1 Compute the number of ordered pairs (x, y) of positive integers satisfying $x^2 - 8x + y^2 + 4y = 5$.

Relay 2-2 Let $T = TNYWR$ and let $k = 21 + 2T$. Compute the largest integer n such that $2n^2 - kn + 77$ is a positive prime number.

Relay 2-3 Let $T = TNYWR$. In triangle ABC, $BC = T$ and $\mathrm{m}\angle B = 30^\circ$. Compute the number of integer values of AC for which there are two possible values for side length AB.

10 Relay Answers

Answer 1-1 $\frac{1}{2}$

Answer 1-2 10

Answer 1-3 109890

Answer 2-1 4

Answer 2-2 12

Answer 2-3 5

11 Relay Solutions

Relay 1-1 A rectangular box has dimensions $8 \times 10 \times 12$. Compute the fraction of the box's volume that is *not* within 1 unit of any of the box's faces.

Solution: Let the box be defined by the product of the intervals on the x, y, and z axes as $[0,8]\times[0,10]\times[0,12]$ with volume $8 \times 10 \times 12$. The set of points inside the box that are not within 1 unit of any face is defined by the product of the intervals $[1,7] \times [1,9] \times [1,11]$ with volume $6 \times 8 \times 10$. This volume is $\frac{6\times8\times10}{8\times10\times12} = \frac{1}{2}$ of the whole box.

Relay 1-2 Let $T = TNYWR$. Compute the largest real solution x to $(\log x)^2 - \log\sqrt{x} = T$.

Solution: Let $u = \log x$. Then the given equation can be rewritten as $u^2 - \frac{1}{2}u - T = 0 \rightarrow 2u^2 - u - 2T = 0$. This quadratic has solutions $u = \frac{1\pm\sqrt{1+16T}}{4}$. As we are looking for the largest real solution for x (and therefore, for u), we want $u = \frac{1+\sqrt{1+16T}}{4} = 1$ when $T = \frac{1}{2}$. Therefore, $x = 10^1 = 10$.

Relay 1-3 Let $T = TNYWR$. Kay has $T+1$ different colors of fingernail polish. Compute the number of ways that Kay can paint the five fingernails on her left hand by using at least three colors and such that no two consecutive fingernails have the same color.

Solution: There are $T+1$ possible colors for the first nail. Each remaining nail may be any color except that of the preceding nail, that is, there are T possible colors. Thus, using at least two colors, there are $(T+1)T^4$ possible colorings. The problem requires that at least three colors be used, so we must subtract the number of colorings that use only two colors. As before, there are $T+1$ possible colors for the first nail and T colors for the second. With only two colors, there are no remaining choices; the colors simply alternate. The answer is therefore $(T+1)T^4 - (T+1)T$, and with $T = 10$, this expression is equal to $110000 - 110 = 109890$.

Relay 2-1 Compute the number of ordered pairs (x, y) of positive integers satisfying $x^2 - 8x + y^2 + 4y = 5$.

Solution: Completing the square twice in x and y, we obtain the equivalent equation $(x-4)^2+(y+2)^2 = 25$, which describes a circle centered at $(4,-2)$ with radius 5. The lattice points on this circle are points 5 units up, down, left, or right of the center, or points 3 units away on one axis and 4 units away on the other. Because the center is below the x-axis, we know that y must increase by at least 2 units; x cannot decrease by 4 or more units if it is to remain positive. Thus, we have:

$$\begin{aligned}
(x,y) &= (4,-2)+(-3,4) = (1,2)\\
(x,y) &= (4,-2)+(0,5) = (4,3)\\
(x,y) &= (4,-2)+(3,4) = (7,2)\\
(x,y) &= (4,-2)+(4,3) = (8,1).
\end{aligned}$$

There are 4 such ordered pairs.

Relay 2-2 Let $T = TNYWR$ and let $k = 21+2T$. Compute the largest value of n such that $2n^2 - kn + 77$ is prime.

Solution: If k is positive, there are only four possible factorizations of $2n^2 - kn + 77$ over the integers, namely

$$\begin{aligned}
(2n-77)(n-1) &= 2n^2 - 79n + 77\\
(2n-1)(n-77) &= 2n^2 - 145n + 77\\
(2n-11)(n-7) &= 2n^2 - 25n + 77\\
(2n-7)(n-11) &= 2n^2 - 29n + 77.
\end{aligned}$$

Because $T = 4$, $k = 29$, and so the last factorization is the correct one. Because $2n-7$ and $n-11$ are both integers, in order for their product to be prime, one factor must equal 1 or -1, so $n = 3, 4, 10$, or 12. Checking these possibilities from the greatest downward, $n = 12$ produces $17 \cdot 1 = 17$, which is prime. So the answer is 12.

Relay 2-3 Let $T = TNYWR$. In triangle ABC, $BC = T$ and $m\angle B = 30°$. Compute the number of integer values of AC for which there are two possible values for side length AB?

Solution: By the Law of Cosines, $(AC)^2 = T^2 + (AB)^2 - 2T(AB)\cos 30° \rightarrow (AB)^2 - 2T\cos 30°(AB) + (T^2 - (AC)^2) = 0$. This quadratic in AB has two positive solutions when the discriminant and product of the roots are both positive. Thus $(2T\cos 30°)^2 - 4(T^2 - (AC)^2) > 0$, and $(T^2 - (AC)^2) > 0$. The second inequality implies that $AC < T$. The first inequality simplifies to $4(AC)^2 - T^2 > 0$, so $T/2 < AC$. Since $T = 12$, we have that $6 < AC < 12$, giving five integral values for AC.

12 Tiebreaker Problems

Problem 1. In $\triangle ABC$, D is on $\overline{AC}$ so that $\overline{BD}$ is the angle bisector of $\angle B$. Point E is on $\overline{AB}$ and $\overline{CE}$ intersects $\overline{BD}$ at P. Quadrilateral $BCDE$ is cyclic, $BP = 12$ and $PE = 4$. Compute the ratio $\frac{AC}{AE}$.

Problem 2. Complete the following "cross-number puzzle", where each "Across" answer represents a four-digit number, and each "Down" answer represents a three-digit number. No answer begins with the digit 0. You must write your answer in the blank 3×4 grid below.

Across:

1. A B C D is the cube of the sum of the digits in the answer to 1 Down.
5. From left to right, the digits in E F G H are strictly decreasing.
6. From left to right, the digits in I J K L are strictly decreasing.

Down:

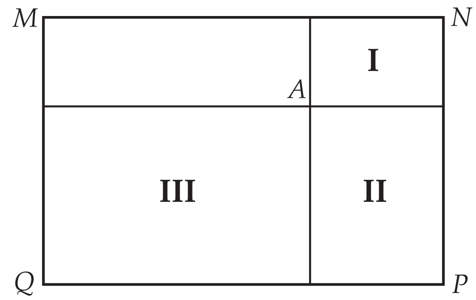
1. A E I is a perfect fourth power.
2. B F J is a perfect square.
3. The digits in C G K form a geometric progression.
4. D H L has a two-digit prime factor.

1 A	2 B	3 C	4 D
5 E	F	G	H
6 I	J	K	L

Answer:

1	2	3	4
5			
6			

Problem 3. In rectangle $MNPQ$, point A lies on $\overline{QN}$. Segments parallel to the rectangle's sides are drawn through point A, dividing the rectangle into four regions. The areas of regions I, II, and III are *integers* in geometric progression. If the area of $MNPQ$ is 2009, compute the maximum possible area of region I.


M
N
I
A
III
II
Q
P

13 Tiebreaker Solutions

Problem 1. In $\triangle ABC$, D is on $\overline{AC}$ so that $\overline{BD}$ is the angle bisector of $\angle B$. Point E is on $\overline{AB}$ and $\overline{CE}$ intersects $\overline{BD}$ at P. Quadrilateral $BCDE$ is cyclic, $BP = 12$ and $PE = 4$. Compute the ratio $\frac{AC}{AE}$.

Answer 1. 3

Solution 1. Let ω denote the circle that circumscribes quadrilateral $BCDE$. Draw in line segment $\overline{DE}$. Note that $\angle DPE$ and $\angle CPB$ are congruent, and $\angle DEC$ and $\angle DBC$ are congruent, since they cut off the same arc of ω. Therefore, $\triangle BCP$ and $\triangle EDP$ are similar. Thus $\frac{BC}{DE} = \frac{BP}{EP} = \frac{12}{4} = 3$.

Because $\angle BCE$ and $\angle BDE$ cut off the same arc of ω, these angles are congruent. Let α be the measure of these angles. Similarly, $\angle DCE$ and $\angle DBE$ cut off the same arc of ω. Let β be the measure of these angles. Since BD is an angle bisector, $\mathrm{m}\angle CBD = \beta$.

Note that $\mathrm{m}\angle ADE = 180° - \mathrm{m}\angle BDE - \mathrm{m}\angle BDC$. It follows that

$$\begin{aligned}
\mathrm{m}\angle ADE &= 180° - \mathrm{m}\angle BDE - (180° - \mathrm{m}\angle CBD - \mathrm{m}\angle BCD) \\
\Rightarrow \mathrm{m}\angle ADE &= 180° - \mathrm{m}\angle BDE - (180° - \mathrm{m}\angle CBD - \mathrm{m}\angle BCE - \mathrm{m}\angle DCE) \\
\Rightarrow \mathrm{m}\angle ADE &= 180° - \alpha - (180° - \beta - \alpha - \beta) \\
\Rightarrow \mathrm{m}\angle ADE &= 2\beta = \mathrm{m}\angle CBD.
\end{aligned}$$

Thus $\angle ADE$ is congruent to $\angle CBD$, and it follows that $\triangle ADE$ is similar to $\triangle ABC$. Hence $\frac{BC}{DE} = \frac{AC}{AE}$, and by substituting in given values, we have $\frac{AC}{AE} = 3$.

Problem 2. Complete the following "cross-number puzzle", where each "Across" answer represents a four-digit number, and each "Down" answer represents a three-digit number. No answer begins with the digit 0. You must write your answer in the blank 3×4 grid below.

Across:
1. A B C D is the cube of the sum of the digits in the answer to 1 Down.
5. From left to right, the digits in E F G H are strictly decreasing.
6. From left to right, the digits in I J K L are strictly decreasing.

Down:
1. A E I is a perfect fourth power.
2. B F J is a perfect square.
3. The digits in C G K form a geometric progression.
4. D H L has a two-digit prime factor.

1 A	2 B	3 C	4 D
5 E	F	G	H
6 I	J	K	L

Answer 2.

(1) 2	(2) 1	(3) 9	(4) 7
(5) 5	4	3	1
(6) 6	4	1	0

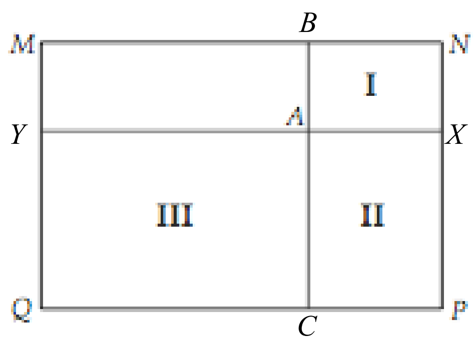
Solution 2. From 1 Down, $\underline{\mathsf{A}}\,\underline{\mathsf{E}}\,\underline{\mathsf{I}} = \mathsf{256}$ or $\mathsf{625}$, either of which make $\underline{\mathsf{A}}\,\underline{\mathsf{B}}\,\underline{\mathsf{C}}\,\underline{\mathsf{D}} = \mathsf{2197}$, so $\underline{\mathsf{A}}\,\underline{\mathsf{E}}\,\underline{\mathsf{I}} = \mathsf{256}$.

From $\underline{\mathsf{B}} = \mathsf{1}$ together with 2 Down, $\underline{\mathsf{B}}\,\underline{\mathsf{F}}\,\underline{\mathsf{J}} = \mathsf{121}$ or $\mathsf{144}$. But $\underline{\mathsf{J}} = \mathsf{1}$ does not work because then 6 Across could not be satisfied. Therefore $\underline{\mathsf{B}}\,\underline{\mathsf{F}}\,\underline{\mathsf{J}} = \mathsf{144}$.

From $\underline{\mathsf{C}} = \mathsf{9}$ together with 5 Across and 3 Down, we have $\underline{\mathsf{C}}\,\underline{\mathsf{G}}\,\underline{\mathsf{K}} = \mathsf{931}$.

From $\underline{\mathsf{D}} = \mathsf{7}$ together with 5 and 6 Across, we get $\underline{\mathsf{D}}\,\underline{\mathsf{H}}\,\underline{\mathsf{L}} = \mathsf{720}$ or $\mathsf{710}$, but only $\mathsf{710}$ has a two-digit prime factor.

Problem 3. In rectangle $MNPQ$, point A lies on $\overline{QN}$. Segments parallel to the rectangle's sides are drawn through point A, dividing the rectangle into four regions. The areas of regions I, II, and III are *integers* in geometric progression. If the area of $MNPQ$ is 2009, compute the maximum possible area of region I.



Answer 3. 1476

Solution 3. Because A is on diagonal $\overline{NQ}$, rectangles $NXAB$ and $ACQY$ are similar. Thus $\frac{AB}{AX} = \frac{QY}{QC} = \frac{AC}{AY} \Rightarrow AB \cdot AY = AC \cdot AX$. Therefore, we have $2009 = [\mathrm{I}] + 2[\mathrm{II}] + [\mathrm{III}]$.

Let the common ratio of the geometric progression be $\frac{p}{q}$, where p and q are relatively prime positive integers (q may equal 1). Then $[\mathrm{I}]$ must be some integer multiple of q^2, which we will call aq^2. This gives $[\mathrm{II}] = apq$ and $[\mathrm{III}] = ap^2$. By factoring, we get

$$2009 = aq^2 + 2apq + ap^2 \Rightarrow 7^2 \cdot 41 = a(p+q)^2.$$

Thus we must have $p + q = 7$ and $a = 41$. Since $[\mathrm{I}] = aq^2$ and $p, q > 0$, the area is maximized when $\frac{p}{q} = \frac{1}{6}$, giving $[\mathrm{I}] = 41 \cdot 36 = 1476$. The areas of the other regions are $246, 246$, and 41.

14 Super-Relay

1. Quadrilateral $ARML$ is a kite with $AR = RM = 5$, $AM = 8$, and $RL = 11$. Compute AL.

2. Let $T = TNYWR$. If $xy = \sqrt{5}$, $yz = 5$, and $xz = T$, compute the positive value of x.

3. Let $T = TNYWR$. In how many ways can T boys and $T + 1$ girls be arranged in a row if all the girls must be standing next to each other?

4. Let $T = TNYWR$. Let $T = TNYWR$. $\triangle ABC$ is on a coordinate plane such that $A = (3, 6)$, $B = (T, 0)$, and $C = (2T - 1, 1 - T)$. Let ℓ be the line containing the altitude to $\overline{BC}$. Compute the y-intercept of ℓ.

5. Let $T = TNYWR$. In triangle ABC, $AB = AC - 2 = T$, and $m\angle A = 60°$. Compute BC^2.

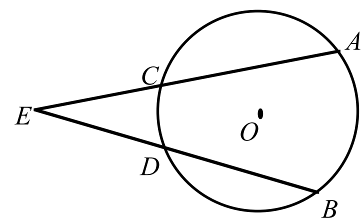
6. Let $T = TNYWR$. Let $\mathcal{S}_1$ denote the arithmetic sequence $0, \frac{1}{4}, \frac{1}{2}, \ldots$, and let $\mathcal{S}_2$ denote the arithmetic sequence $0, \frac{1}{6}, \frac{1}{3}, \ldots$. Compute the T^{th} smallest number that occurs in both sequences $\mathcal{S}_1$ and $\mathcal{S}_2$.

7. Let $T = TNYWR$. An integer n is randomly selected from the set $\{1, 2, 3, \ldots, 2T\}$. Compute the probability that the integer $|n^3 - 7n^2 + 13n - 6|$ is a prime number.

8. Let A be the number you will receive from position 7, and let B be the number you will receive from position 9. In $\frac{1}{A}$ minutes, 20 frogs can eat 1800 flies. At this rate, in $\frac{1}{B}$ minutes, how many flies will 15 frogs be able to eat?

9. Let $T = TNYWR$. If $|T| - 1 + 3i = \frac{1}{z}$, compute the sum of the real and imaginary parts of z.

10. Let $T = TNYWR$. In circle O, diagrammed at right, minor arc $\widehat{AB}$ measures $\frac{T}{4}$ degrees. If $m\angle OAC = 10°$ and $m\angle OBD = 5°$, compute the degree measure of $\angle AEB$. Just pass the number without the units.



11. Let $T = TNYWR$. Ann spends 80 seconds climbing up a T meter rope at a constant speed, and she spends 70 seconds climbing down the same rope at a constant speed (different from her upward speed). Ann begins climbing up and down the rope repeatedly, and she does not pause after climbing the length of the rope. After T minutes, how many meters will Ann have climbed in either direction?

12. Let $T = TNYWR$. Simplify $2^{\log_4 T}/2^{\log_{16} 64}$.

13. Let $T = TNYWR$. Let $P(x) = x^2 + Tx + 800$, and let r_1 and r_2 be the roots of $P(x)$. The polynomial $Q(x)$ is quadratic, it has leading coefficient 1, and it has roots $r_1 + 1$ and $r_2 + 1$. Find the sum of the coefficients of $Q(x)$.

14. Let $T = TNYWR$. Equilateral triangle ABC is given with side length T. Points D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Point F lies in space such that $\triangle DEF$ is equilateral and $\triangle DEF$ lies in a plane perpendicular to the plane containing $\triangle ABC$. Compute the volume of tetrahedron $ABCF$.

15. In triangle ABC, $AB = 5, AC = 6$, and $\tan \angle BAC = -\frac{4}{3}$. Compute the area of $\triangle ABC$.

15 Super-Relay Answers

Answer to the relay: 3750

1. $4\sqrt{5}$ 2. 2 3. 36 4. 3 5. 19 6. 9 7. $\frac{1}{9}$ 8. 3750
9. $\frac{1}{25}$ 10. 5 11. 80 12. 10 13. 800 14. 108 15. 12

16 Super-Relay Solutions

1. Let K be the midpoint of $\overline{AM}$. Then $AK = KM = 8/2 = 4$, $RK = \sqrt{5^2 - 4^2} = 3$, and $KL = 11 - 3 = 8$. Thus $AL = \sqrt{AK^2 + KL^2} = \sqrt{4^2 + 8^2} = 4\sqrt{5}$.

2. Multiply the three given equations to obtain $x^2y^2z^2 = 5T\sqrt{5}$. Thus $xyz = \pm\sqrt[4]{125T^2}$, and the positive value of x is $x = xyz/yz = \sqrt[4]{125T^2}/5 = \sqrt[4]{T^2/5}$. With $T = 4\sqrt{5}$, we have $x = 2$.

3. First choose the position of the first girl, starting from the left. There are $T + 1$ possible positions, and then the positions for the girls are all determined. There are $(T + 1)!$ ways to arrange the girls, and there are $T!$ ways to arrange the boys, for a total of $(T + 1) \cdot (T + 1)! \cdot T! = ((T + 1)!)^2$ arrangements. With $T = 2$, the answer is 36.

4. The slope of $\overleftrightarrow{BC}$ is $\frac{(1-T)-0}{(2T-1)-T} = -1$, and since ℓ is perpendicular to $\overleftrightarrow{BC}$, the slope of ℓ is 1. Because ℓ passes through $A = (3, 6)$, the equation of ℓ is $y = x + 3$, and its y-intercept is 3 (independent of T).

5. By the Law of Cosines, $BC^2 = AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos A = T^2 + (T+2)^2 - 2 \cdot T \cdot (T+2) \cdot \frac{1}{2} = T^2 + 2T + 4$. With $T = 3$, the answer is 19.

6. $\mathcal{S}_1$ consists of all numbers of the form $\frac{n}{4}$, and $\mathcal{S}_2$ consists of all numbers of the form $\frac{n}{6}$, where n is a nonnegative integer. Since $\gcd(4, 6) = 2$, the numbers that are in both sequences are of the form $\frac{n}{2}$, and the T^{th} smallest such number is $\frac{T-1}{2}$. With $T = 19$, the answer is 9.

7. Let $P(n) = n^3 - 7n^2 + 13n - 6$, and note that $P(n) = (n - 2)(n^2 - 5n + 3)$. Thus $|P(n)|$ is prime if either $|n-2| = 1$ and $|n^2 - 5n + 3|$ is prime or if $|n^2 - 5n + 3| = 1$ and $|n - 2|$ is prime. Solving $|n - 2| = 1$ gives $n = 1$ or 3, and solving $|n^2 - 5n + 3| = 1$ gives $n = 1$ or 4 or $\frac{5\pm\sqrt{17}}{2}$. Note that $P(1) = 1, P(3) = -3$, and $P(4) = -2$. Thus $|P(n)|$ is prime only when n is 3 or 4, and if $T \geq 2$, then the desired probability is $\frac{2}{2T} = \frac{1}{T}$. With $T = 9$, the answer is $\frac{1}{9}$.

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15. Let $s = \sin\angle BAC$. Then $s > 0$ and $\frac{s}{-\sqrt{1-s^2}} = -\frac{4}{3}$, which gives $s = \frac{4}{5}$. The area of triangle ABC is therefore $\frac{1}{2} \cdot AB \cdot AC \cdot \sin\angle BAC = \frac{1}{2} \cdot 5 \cdot 6 \cdot \frac{4}{5} = 12$.

14. The volume of tetrahedron $ABCF$ is one-third the area of $\triangle ABC$ times the distance from F to $\triangle ABC$. Since D and E are midpoints, $DE = \frac{BC}{2} = \frac{T}{2}$, and the distance from F to $\triangle ABC$ is $\frac{T\sqrt{3}}{4}$. Thus the volume of $ABCF$ is $\frac{1}{3} \cdot \frac{T^2\sqrt{3}}{4} \cdot \frac{T\sqrt{3}}{4} = \frac{T^3}{16}$. With $T = 12$, the answer is 108.

13. Let $Q(x) = x^2 + Ax + B$. Then $A = -(r_1 + 1 + r_2 + 1)$ and $B = (r_1 + 1)(r_2 + 1)$. Thus the sum of the coefficients of $Q(x)$ is $1 + (-r_1 - r_2 - 2) + (r_1r_2 + r_1 + r_2 + 1) = r_1r_2$. Note that $T = -(r_1 + r_2)$ and $800 = r_1r_2$, so the answer is 800 (independent of T). [Note: With $T = 108$, $\{r_1, r_2\} = \{-8, -100\}$.]

12. Note that $2^{\log_4 T} = 4^{(\frac{1}{2}\log_4 T)} = 4^{\log_4 T^{\frac{1}{2}}} = \sqrt{T}$. Letting $\log_{16} 64 = x$, we see that $2^{4x} = 2^6$, thus $x = \frac{3}{2}$, and $2^x = \sqrt{8}$. Thus the given expression equals $\sqrt{\frac{T}{8}}$, and with $T = 800$, this is equal to 10.

11. In 150 seconds (or 2.5 minutes), Ann climbs up and down the entire rope. Thus in T minutes, she makes $\lfloor\frac{T}{2.5}\rfloor$ round trips, and therefore climbs $2T\lfloor\frac{T}{2.5}\rfloor$ meters. After making all her round trips, there are $t = 60(T - 2.5\lfloor\frac{T}{2.5}\rfloor)$ seconds remaining. If $t \leq 80$, then the remaining distance climbed is $T \cdot \frac{t}{80}$ meters, and if $t > 80$, then the distance climbed is $T + T \cdot \left(\frac{t-80}{70}\right)$ meters. In general, the total distance in meters that Ann climbs is $2T\lfloor\frac{T}{2.5}\rfloor + T \cdot \min\left(1, \frac{60\left(T-2.5\lfloor\frac{T}{2.5}\rfloor\right)}{80}\right) + T \cdot \max\left(0, \frac{60\left(T-2.5\lfloor\frac{T}{2.5}\rfloor\right)-80}{70}\right)$. With $T = 10$, Ann makes exactly 4 round trips, and therefore climbs a total of $4 \cdot 2 \cdot 10 = 80$ meters.

10. Note that $\mathrm{m}\angle AEB = \frac{1}{2}(\mathrm{m}\overset{\frown}{AB} - m\,\overset{\frown}{CD}) = \frac{1}{2}(\mathrm{m}\overset{\frown}{AB} - \mathrm{m}\angle COD)$. Also note that $\mathrm{m}\angle COD = 360^\circ - (\mathrm{m}\angle AOC + \mathrm{m}\angle BOD + \mathrm{m}\angle AOB) = 360^\circ - (180^\circ - 2\mathrm{m}\angle OAC) - (180^\circ - 2\mathrm{m}\angle OBD) - \mathrm{m}\overset{\frown}{AB} = 2(\mathrm{m}\angle OAC + \mathrm{m}\angle OBD) - \mathrm{m}\overset{\frown}{AB}$. Thus $\mathrm{m}\angle AEB = \mathrm{m}\overset{\frown}{AB} - \mathrm{m}\angle OAC - \mathrm{m}\angle OBD = \frac{T}{4} - 10^\circ - 5^\circ$, and with $T = 80$, the answer is 5.

9. Let $t = |T|$. Note that $z = \frac{1}{t-1+3i} = \frac{1}{t-1+3i} \cdot \frac{t-1-3i}{t-1-3i} = \frac{t-1-3i}{t^2-2t+10}$. Thus the sum of the real and imaginary parts of z is $\frac{t-1}{t^2-2t+10} + \frac{-3}{t^2-2t+10} = \frac{|T|-4}{|T|^2-2|T|+10}$. With $T = 5$, the answer is $\frac{1}{25}$.

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8. In $\frac{1}{A}$ minutes, 1 frog can eat $1800/20 = 90$ flies; thus in $\frac{1}{B}$ minutes, 1 frog can eat $\frac{A}{B} \cdot 90$ flies. Thus in $\frac{1}{B}$ minutes, 15 frogs can eat $15 \cdot 90 \cdot \frac{A}{B}$ flies. With $A = \frac{1}{9}$ and $B = \frac{1}{25}$, this simplifies to $15 \cdot 250 = 3750$.