12^{th} Annual Harvard-MIT Math Tournament

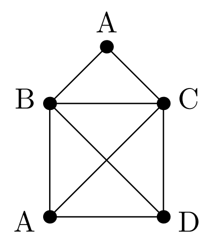
Saturday 21 February 2009

Team Round - Division A

A *graph* consists of a set of vertices and a set of edges connecting pairs of vertices. If vertices u and v are connected by an edge, we say they are *adjacent*. The *degree* of a vertex is the number of vertices adjacent to it.

A *coloring* of a graph is an assignment of a color to each vertex of the graph. A coloring is *good* if no two adjacent vertices are the same color. The *chromatic number* of a graph is the minimum number of colors needed in any good coloring of that graph.

For example, the chromatic number of the following graph is 4. The good coloring shown below uses four colors, and there is no good coloring using only three colors because four of the vertices are mutually adjacent.



1. [**8**] Let $n \geq 3$ be a positive integer. A *triangulation* of a convex n-gon is a set of $n-3$ of its diagonals which do not intersect in the interior of the polygon. Along with the n sides, these diagonals separate the polygon into $n-2$ disjoint triangles. Any triangulation can be viewed as a graph: the vertices of the graph are the corners of the polygon, and the n sides and $n-3$ diagonals are the edges.

   For a fixed n-gon, different triangulations correspond to different graphs. Prove that all of these graphs have the same chromatic number.

2. (a) [**4**] Let P be a graph with one vertex v_n for each positive integer n. If $a < b$, then an edge connects vertices v_a and v_b if and only if $\frac{b}{a}$ is a prime number. What is the chromatic number of P? Prove your answer.

   (b) [**6**] Let T be a graph with one vertex v_n for every integer n. An edge connects v_a and v_b if $|a-b|$ is a power of two. What is the chromatic number of T? Prove your answer.

3. A graph is *finite* if it has a finite number of vertices.

   (a) [**6**] Let G be a finite graph in which every vertex has degree k. Prove that the chromatic number of G is at most $k+1$.

   (b) [**10**] In terms of n, what is the minimum number of edges a finite graph with chromatic number n could have? Prove your answer.

4. A k-*clique* of a graph is a set of k vertices such that all pairs of vertices in the clique are adjacent.

   (a) [**6**] Find a graph with chromatic number 3 that does not contain any 3-cliques.

(b) [**10**] Prove that, for all $n > 3$, there exists a graph with chromatic number n that does not contain any n-cliques.

5. The *size* of a finite graph is the number of vertices in the graph.

   (a) [**15**] Show that, for any $n > 2$, and any positive integer N, there are finite graphs with size at least N and with chromatic number n such that removing any vertex (and all its incident edges) from the graph decreases its chromatic number.

   (b) [**15**] Show that, for any positive integers n and r, there exists a positive integer N such that for any finite graph having size at least N and chromatic number equal to n, it is possible to remove r vertices (and all their incident edges) in such a way that the remaining vertices form a graph with chromatic number at least $n - 1$.

6. For any set of graphs $G_1, G_2, \ldots, G_n$ all having the same set of vertices V, define their *overlap*, denoted $G_1 \cup G_2 \cup \cdots \cup G_n$, to be the graph having vertex set V for which two vertices are adjacent in the overlap if and only if they are adjacent in at least one of the graphs G_i. In other words, the edge set of the overlap of the graphs G_i is the union of their edge sets.

   (a) [**10**] Let G and H be graphs having the same vertex set and let a be the chromatic number of G and b the chromatic number of H. Find, in terms of a and b, the largest possible chromatic number of $G \cup H$. Prove your answer.

   (b) [**10**] Suppose G is a graph with chromatic number n. Suppose there exist k graphs $G_1, G_2, \ldots, G_k$ having the same vertex set as G such that $G_1 \cup G_2 \cup \cdots \cup G_k = G$ and each G_i has chromatic number at most 2. Show that $k \geq \lceil \log_2(n) \rceil$.

7. [**20**] Let n be a positive integer. Let V_n be the set of all sequences of 0's and 1's of length n. Define G_n to be the graph having vertex set V_n, such that two sequences are adjacent in G_n if and only if they differ in either 1 or 2 places. For instance, if $n = 3$, the sequences $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$ are mutually adjacent, but $(1, 0, 0)$ is not adjacent to $(0, 1, 1)$.

   Show that, if $n + 1$ is not a power of 2, then the chromatic number of G_n is at least $n + 2$.

8. [**30**] Two colorings are *distinct* if there is no way to relabel the colors to transform one into the other. Equivalently, they are distinct if and only if there is some pair of vertices which are the same color in one coloring but different colors in the other. For what pairs (n, k) of positive integers does there exist a finite graph with chromatic number n which has exactly k distinct good colorings using n colors?