2009 Marywood Mathematics Contest

Level II

Sponsored by

SEMI-GROUP

The Student Mathematics Club of

Marywood University

March 21, 2009

Directions:

1. This exam consists of 40 questions on 7 pages. Please check to make sure that you have all the pages.

2. Allot your time accordingly. This is a 60-minute test. Do not spend too much time on any one problem. If a question seems to be too difficult, make your best possible guess. Your score will be the number of correct responses.

3. On the scantron form provided for you, darken in the space corresponding to the correct answer. Please mark all answers carefully and erase completely when changing an answer. Mark **only one answer** for each question. Only those answers on the answer sheet will be counted.

4. There is a sheet of blank paper on the last page which you can (carefully) tear off and use as scratch paper. You may also use the back of the pages.

5. NOTE: **In order to ensure uniformity, proctors are NOT allowed to answer any questions pertaining to specific problem content.**

Please do NOT open the test until you are told to do so.

1. $6^6 + 6^6 + 6^6 + 6^6 + 6^6 + 6^6 =$

   A. 6^6 **B.** 36^6 **C.** 6^{36} **D.** 36^{36} **E.** 6^7

2. If $3(4x + 5\pi) = P$, then $6(8x + 10\pi) =$

   A. $2P$ **B.** $4P$ **C.** $6P$ **D.** $8P$ **E.** $18P$

3. If $m > 0$ and the points $(m, 3)$ and $(1, m)$ lie on a line with slope m, then $m =$

   A. 1   **B.** $\sqrt{2}$ **C.** $\sqrt{3}$ **D.** 2   **E.** $\sqrt{5}$

4. If a, b, and c are positive integers and a and b are odd, then $3^a + (b - 1)^2 c$ is

   A. odd for all choices of c.   **B.** even for all choices of c.

   C. odd, if c is even; even, if c is odd.   **D.** odd, if c is odd; even, if c is even.

   E. odd, if c is not a multiple of 3; even if c is a multiple of 3.

5. A lily pad grows so that each day it doubles its size (area). On the 20th day of its life, it completely covers the pond. On what day of its life was the pond half covered?

   A. 5th   **B.** 10th   **C.** 15th   **D.** 19th   **E.** None of these.

6. What is the diameter (in cm) of the circle whose area (in cm^2) and circumference (in cm) have the same numerical value?

   A. 1 cm   **B.** 2 cm   **C.** 3 cm   **D.** π cm   **E.** 4 cm

7. If $x > y > 0$, then $\dfrac{x^y \, y^x}{y^y \, x^x} =$

   A. $(x - y)^{y/x}$ **B.** $\left(\dfrac{x}{y}\right)^{x-y}$ **C.** 1   **D.** $\left(\dfrac{x}{y}\right)^{y-x}$ **E.** $(x - y)^{x/y}$

8. The ratio of w to x is $4 : 3$, of y to z is $3 : 2$ and of z to x is $1 : 6$. What is the ratio of w to y?

   A. $1 : 3$ **B.** $16 : 3$ **C.** $20 : 3$ **D.** $27 : 4$ **E.** $12 : 1$

9. For all non-zero real numbers x and y such that $x - y = xy$, $\dfrac{1}{x} - \dfrac{1}{y} =$

A. $\dfrac{1}{xy}$ **B.** $\dfrac{1}{x-y}$ **C.** 1 **D.** -1 **E.** $y - x$

10. A square floor is tiled with congruent square tiles. The tiles on the two diagonals of the floor are black. The rest of the tiles are white. If there are 101 black tiles, then the total number of tiles is
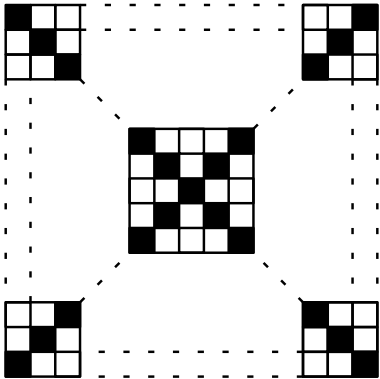
A. 121

B. 625

C. 676

D. 2500

E. 2601

11. SuperSneaky Department Store is having a 50% "plus" 40% off sale on everything they have in store, which entitles you to 50% off the original price first and then a 40% off this discounted price. How much does a coat originally priced at $150 cost now?

A. $15   **B.** $30   **C.** $45   **D.** $60   **E.** None of these.

12. The number of positive integers k for which the equation $kx - 12 = 3k$ has an integer solution for x is

**A.** 3   **B.** 4   **C.** 5   **D.** 6   **E.** 7

13. The radii of the three circles below are in the ratio $1 : 2 : 3$. What is the probability that a random shot that hits the target will hit inside the inner most circle?
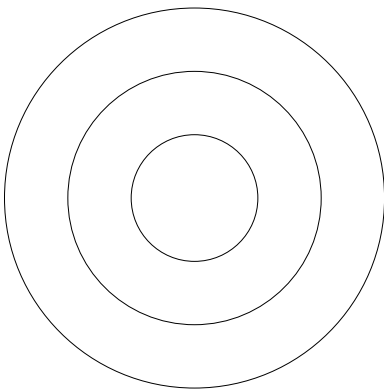
A. $\dfrac{1}{3}$ **B.** $\dfrac{1}{6}$

C. $\dfrac{1}{9}$ **D.** $\dfrac{3}{16}$

E. None of these.

14. Five equilateral triangle, each with side $2\sqrt{3}$, are arrange so they are all on the side of a line containing one side of each. Along this line, the midpoint of the base of one triangle is a vertex of the next. The area of the region of the plane that is covered by the union of the five triangular regions is

A. 10 B. 12

C. 15 D. $10\sqrt{3}$

E. $12\sqrt{3}$
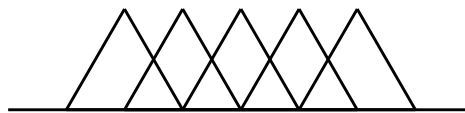
15. Which of the following equations represents a parabola with y-intercept equal to -6 and x-intercepts equal to -1 and 2?

A. $y = x^2 - x - 2$ B. $y = 3x^2 - 3x - 6$ C. $y = x^2 + 5x - 6$

D. $y = x^2 - x - 6$ E. None of these.

16. Let $y = mx + b$ be the image when the line $x - 3y + 11 = 0$ is reflected across the x-axis. The value of $m + b$ is

A. -6 B. -5 C. -4 D. -3 E. -2

17. How many pairs of positive integers (a, b) with $a + b \le 100$ satisfy the equation

$$\frac{a + b^{-1}}{a^{-1} + b} = 13?$$

A. 1   B. 5   C. 7   D. 9   E. 13

18. Which of the following equations have the same graph?

I. $y = x - 2$ II. $y = \dfrac{x^2 - 4}{x + 2}$ III. $(x + 2)y = x^2 - 4$

A. I and II only.   B. I and III only.

C. II and III only.   D. I, II, and III.

E. None. All the equations have different graphs.

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19. Consider the sequence defined recursively by $u_1 = a$ (where a is any positive number), and $u_{n+1} = \dfrac{-1}{u_n + 1}$, $n = 1, 2, 3, \ldots$. For which of the following values of n must $u_n = a$?

A. 14   **B.** 15   **C.** 16   **D.** 17   **E.** 18

20. An urn is filled with coins and beads, all of which are either silver or gold. Twenty percent of the objects in the urn are beads. Forty percent of the coins in the urn are silver. What percent of the objects in the urn are gold coins?

A. 40%   **B.** 48%   **C.** 52%   **D.** 60%   **E.** 80%

21. Let $i = \sqrt{-1}$. Define a sequence of complex numbers by $z_1 = 0$, $z_{n+1} = z_n^2 + i$ for $n \geq 1$. In the complex plane, how far from the origin is z_{111}?

A. 1   **B.** $\sqrt{2}$ **C.** $\sqrt{3}$ **D.** $\sqrt{110}$ **E.** $\sqrt{2^{55}}$

22. If $\dfrac{y}{x-z} = \dfrac{x+y}{z} = \dfrac{x}{y}$ for three positive numbers x, y, and z, all different, then $\dfrac{x}{y} =$

A. $\dfrac{1}{2}$ **B.** $\dfrac{3}{5}$ **C.** $\dfrac{2}{3}$ **D.** $\dfrac{5}{3}$ **E.** 2

23. Given that $f(x) = 3x + 4$ and $f(g(x)) = x$, find an expression for $g(x)$.

A. $g(x) = \dfrac{1}{3}x - 4$ **B.** $g(x) = 3x - 4$ **C.** $g(x) = 4x - 3$

D. $g(x) = \dfrac{1}{3}x + 4$ **E.** None of these.

24. The center C and the radius r of the circle represented by $x^2 + y^2 + 10x - 14y = -38$ are

A. $C : (-5, 7)$, $r = 6$.   **B.** $C : (5, -7)$, $r = 36$.   **C.** $C : (-5, -7)$, $r = 6$.

D. $C : (-7, 5)$, $r = 6$.   **E.** None of these.

25. Express the following product as a reduced fraction.

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{2009}\right)$$

A. $\dfrac{2008}{2009}$ **B.** $\dfrac{1}{2009}$ **C.** $\dfrac{143}{287}$ **D.** $\dfrac{20}{41}$ **E.** None of these.

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26. Joe walked to the crater at a speed of 1.5 mph. He returned along the same path by horseback at 9 mph. The roudtrip took 21 hours. How many miles is it to the crater from where he originally was?

A. 27   **B.** 3   **C.** 18   **D.** 110.25   **E.** 7

27. The vertex of the parabola represented by $y^2 - 8x - 6y + 41 = 0$ is

A. $(-4, -3)$ **B.** $(-4, 3)$ **C.** $(4, 3)$ **D.** $(4, -3)$ **E.** None of these.

28. Sparrows and pigeons sit on a fence. When 5 sparrows leave, there remain 2 pigeons for every sparrow. Then 25 pigeons leave and the ratio of sparrows to pigeons becomes $3 : 1$. What is the original number of birds?

A. 50   **B.** 48   **C.** 75   **D.** 30   **E.** None of these.

29. Let $[x]$ be the greatest integer that is less than or equal to x and $f(x) = [x]^2 - 2[x]$. Find $f(2\pi)$.

A. 32   **B.** 16   **C.** 24   **D.** 12   **E.** None of these.

30. If $(-2, 8), (8, -15)$ and $(-6, k)$ are collinear points, then the value of k is

A. 13.5   **B.** 14.2   **C.** 18.4   **D.** 17.2   **E.** None of these.

31. Given $f(x) = (x + 6)^3 + 1$, then $f^{-1}(x) =$

A. $\sqrt[3]{x - 1} - 6$.   **B.** $\sqrt[3]{4x - 5} - 6$.   **C.** $(4x - 7)^3 - 3$

D. $\sqrt{5x^2 + 9} - 3$.   **E.** None of these.

32. The degree measure (to the nearest minute) of the central angle having intercepted arc measuring 15 ft in a circle of diameter 19 ft is:

A. $15°18'$ **B.** $90°28'$ **C.** $62°15'$ **D.** $78°32'$ **E.** None of these.

33. Given that the terminal side of the angle θ is in Quadrant IV with its initial side being the positive x-axis and $\csc \theta = -\dfrac{12}{7}$, what is the value of $\cot \theta$?

A. $\dfrac{\sqrt{29}}{7}$ **B.** $-\dfrac{\sqrt{95}}{7}$ **C.** $-\dfrac{\sqrt{65}}{7}$ **D.** $\dfrac{\sqrt{85}}{7}$ **E.** None of these.

34. The ratio of the radii of two concentric circles is 1:3. If $\overline{AC}$ is a diameter of the larger circle, $\overline{BC}$ is a chord of the larger circle that is tangent to the smaller circle, and $AB = 12$, then the radius of the large circle is
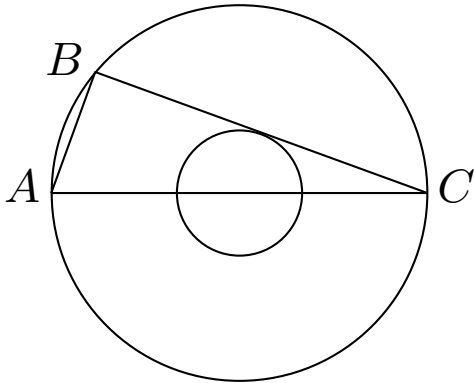

A. 13

B. 18

C. 21

D. 24

E. 26

35. Given the rectangle $ABCD$ below and E bisects DC and F bisects AD. Find the area of the quadrilateral $BEDF$.

A. 7.5 in^2

B. 6.5 in^2

C. 6.75 in^2

D. 7.75 in^2

E. None of these.
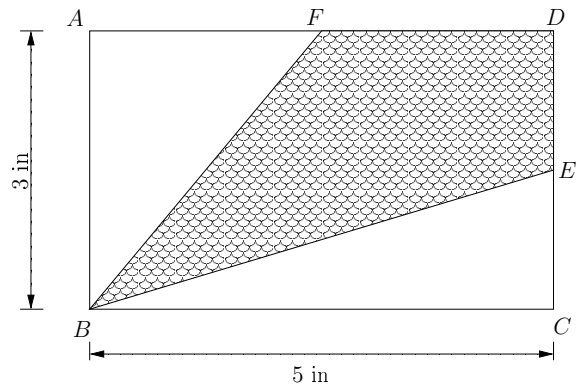
36. Beula, a lovable cow, is located 1 mile north of the Pasture River along the fence line. She wants to get to her barn to take a nap. Her barn is located 3 miles east of the fence line and 2 miles north of the river. She also wants to stop at the river for a drink along the way. Where on the river should she stop in order to minimize her total amount of walking? The answers below are given as coordinates on the x-axis, which is the Pasture River, and we assume the fence is the y-axis.
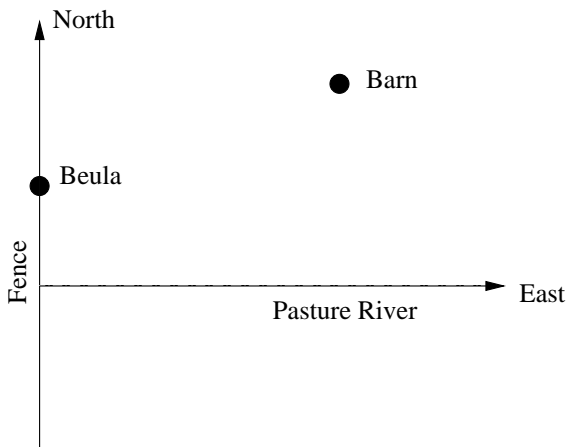
A. $(2, 0)$

B. $(2.5, 0)$

C. $(1, 0)$

D. $(1.5, 0)$

E. None of these.

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37. How many ways can 10 men and 7 women sit in a row so that no two women are next to each other?

A. 17! **B.** $(10!)(17!)$ **C.** $330(10!)(17!)$

D. $120(10!)(17!)$ **E.** None of these.

38. The integers from 200 down to 9 are written consecutively to form the large integer

$$N \; = \; 200199198197\ldots131211109.$$

If 3^k is the highest power of 3 that is a factor of N, then the value of k is

A. 0. **B.** 1. **C.** 2. **D.** 3. **E.** more than 3.

39. The increasing sequence of positive integers a_1, a_2, a_3, ... has the property that $a_{n+2} = a_n + a_{n+1}$ for all $n \geq 1$. If $a_7 = 120$, then a_8 is

A. 128 **B.** 168 **C.** 193 **D.** 194 **E.** 210

40. For each vertex of a solid cube, consider the tetrahedron determined by the vertex of the midpoints of the three edges that meet at the vertex. The portion of the cube that remains when these eight tetrahedra are cut away is called a cuboctahedron. The ratio of the volume of the cuboctahedron to the volume of the original cube is closest to which of the following?

A. 75% **B.** 78% **C.** 81% **D.** 84% **E.** 87%

There are no problems on this page and it is intended to be used as scratch paper.